BASICS OF MATHEMATICS

AND PHYSICS

FOR PRE-UNIVERSITY COURSE

Minsk BSMU 2017

МИНИСТЕРСТВО ЗДРАВООХРАНЕНИЯ РЕСПУБЛИКИ БЕЛАРУСЬ
БЕЛОРУССКИЙ ГОСУДАРСТВЕННЫЙ МЕДИЦИНСКИЙ УНИВЕРСИТЕТ
КАФЕДРА МЕДИЦИНСКОЙ И БИОЛОГИЧЕСКОЙ ФИЗИКИ

ОСНОВЫ МАТЕМАТИКИ И ФИЗИКИ ДЛЯ ПОДГОТОВИТЕЛЬНОГО ОТДЕЛЕНИЯ

BASICS OF MATHEMATICS AND PHYSICS FOR PRE-UNIVERSITY COURSE

Учебно-методическое пособие

Минск БГМУ 2017

УДК 51+53(075.8)-054.6
ББК 22.1+22.3я73
О-75

Рекомендовано Научно-методическим советом университета в качестве учебно-методического пособия 15.06.2016 г., протокол № 10

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О-75 **Основы** математики и физики для подготовительного отделения = Basics of mathematics and physics for pre-university course : учеб.-метод. пособие / М. В. Гольцев [и др.]. – Минск : БГМУ, 2017. – 224 с.

ISBN 978-985-567-680-6.

Включает некоторые сведения по элементарной математике и основам дифференциального исчисления, а также основные разделы школьного курса физики — механику, молекулярную физику и термодинамику, электричество и магнетизм, оптику, атомную и ядерную физику.

Предназначено для иностранных учащихся подготовительного отделения, обучающихся на английском языке.

УДК 51+53(075.8)-054.6
ББК 22.1+22.3я73

Учебное издание

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Ответственный за выпуск М. В. Гольцев
Переводчики: М. В. Гольцев, Л. В. Кухаренко, О. В. Недзьведь, Н. А. Никоненко
Компьютерная верстка Н. М. Федорцовой

Подписано в печать 01.03.17. Формат 60×84/16. Бумага писчая «Снегурочка». Ризография. Гарнитура «Times». Усл. печ. л. 13,02. Уч.-изд. л. 10,28. Тираж 150 экз. Заказ 117.

Издатель и полиграфическое исполнение: учреждение образования «Белорусский государственный медицинский университет».
Свидетельство о государственной регистрации издателя, изготовителя, распространителя печатных изданий № 1/187 от 18.02.2014.
Ул. Ленинградская, 6, 220006, Минск.

ISBN 978-985-567-680-6

PREFACE

The textbook is intended for the international students of the preparatory departments of institutions of higher education studying physics in English in order to enter the medical universities of the Republic of Belarus.

This textbook contents information on elementary mathematics and the basics of differential calculus, as well as the main sections of a school physics course — mechanics, molecular physics and thermodynamics, electricity and magnetism, optics, atomic and nuclear physics, which are required for the study of medical and biological physics. The problems and examples have been selected for the distinct purpose of illustrating the principles taught in the text and for their practical applications. A list of problems and tests is placed at the end of every topic. They are in sufficient number to permit testing at many points and of a teacher's choice of problems. The order of topics, illustrations, and problems have been also selected with the purpose of leading the student into a clear understanding of the physical phenomena concerning to biology and medicine. Duly made selection of problems and examples, conciseness and simplicity of presentation of different topics contribute to the successful study of the proposed material.

THE BASICS OF ELEMENTARY MATHEMATICS AND DIFFERENTIAL CALCULUS

1. THE BASIC MATHEMATICAL CONCEPTS AND FORMULAS

1.1. FRACTION. OPERATIONS WITH FRACTIONS. EXPONENTS AND RADICALS. FACTORING AND EXPANDING

A fraction is an expression of the following form: $\frac{a}{b}$ (***a*** **over** ***b***), where ***a*** — numerator, ***b*** — denominator.

A proper fraction is one whose numerator is less than denominator. For example, $^1/_2$; $^1/_4$; $^1/_7$; $^2/_3$; $^2/_{100}$ and $^{36}/_{81}$ are proper fractions. An improper fraction is a fraction, whose numerator is equal to or larger than the denominator. Thus, $^{21}/_5$, $^{100}/_{37}$ and $^8/_8$ are improper fractions.

Operations with fractions.

To reduce a fraction to its lowest terms, divide numerator and denominator by their highest common factor (or: measure, or: divisor: $\frac{ar}{br}=\frac{a}{b}$).

To reduce a fraction to higher terms, multiply the numerator and the denominator by the same number: $\frac{a}{b}=\frac{ar}{br}$.

To find the sum (the difference) of two unlike fractions, change them to like fractions (fractions having their least common denominator) and combine the numerators: $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm cb}{bd}$.

To find the product of two fractions, multiply the numerators together and the denominators together: $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

To find the quotient of two fractions, multiply the dividend by the inverted divisor: $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$.

Equivalent fractions are known as proportions:

$$\frac{a}{b}=\frac{c}{d}\Leftrightarrow ad=bc\Leftrightarrow\frac{a}{c}=\frac{b}{d}\Leftrightarrow\frac{c}{a}=\frac{d}{b}\Leftrightarrow\frac{d}{c}=\frac{b}{a}.$$

Exponents and Radicals (Roots).

In the expression $(a^n)=c$ (***a*** to the ***n**-th* power is equal to ***c***) the quantity ***a*** is called the base and ***n*** is the exponent of the power.

A quantity $\boldsymbol{a}$ to the power of $\boldsymbol{m}$ over $\boldsymbol{n}$ is called the $\boldsymbol{n}$-*th* root of $\boldsymbol{a}$ to the $\boldsymbol{m}$-*th* power. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

The following rules are useful in manipulations with exponents and roots.

$a^0 = 1$

$a^1 = a$

$(a^n)(a^m) = a^{n+m}$

$(a^n)(b^n) = (ab)^n$

$(a^n)^m = a^{nm}$

$(a^n)(a^{-n}) = a^0 = 1$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\frac{a^n}{a^m} = (a)^{n-m}$$

$$\frac{1}{a^n} = a^{-n}$$

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Factoring and Expanding.

In many cases one needs the inverse operations — Factoring or Expanding. We can obtain another form of the algebraic expression due to expanding of powers or a product of items and write down the result as a sum of terms.

Table 1.1

Expanding formulas

1	$(a + b)^2 = a^2 + 2ab + b^2$
2	$(a - b)^2 = a^2 - 2ab + b^2$
3	$(a + b) \cdot (a - b) = a^2 - b^2$
4	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
5	$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
6	$(a + b)(a^2 - ab + b^2) = a^3 + b^3$
7	$(a - b)\cdot(a^2 + ab + b^2) = a^3 - b^3$

***Examples*:**

$$\frac{(a+b)^2-(a-b)^2}{ab} = \frac{a^2+2ab+b^2-a^2+2ab-b^2}{ab} = \frac{4ab}{ab} = 4.$$

$$\frac{x}{y-1}+\frac{5}{1-y}=\frac{x}{y-1}-\frac{5}{y-1}=\frac{x-5}{y-1}$$

$$\frac{a}{c-3}-\frac{6}{3-c}=\frac{a}{c-3}+\frac{6}{c-3}=\frac{a+6}{c-3}$$

$$\frac{2m}{m-n}+\frac{2n}{n-m}=\frac{2m}{m-n}-\frac{2n}{m-n}=\frac{2m-2n}{m-n}=\frac{2(m-n)}{m-n}=2$$

$$\frac{5p}{2q-p}+\frac{10q}{p-2q}=\frac{5p}{2q-p}-\frac{10q}{2q-p}=\frac{5p-10q}{2q-p}=\frac{5(p-2q)}{2q-p}=-5$$

$$\frac{a^2+16}{a-4}+\frac{8a}{4-a}=\frac{a^2+16}{a-4}-\frac{8a}{a-4}=\frac{a^2-8a+16}{a-4}=\frac{(a-4)^2}{a-4}=a-4$$

$$\frac{x^2+9y^2}{x-3y}+\frac{6xy}{3y-x}=\frac{x^2+9y^2}{x-3y}-\frac{6xy}{x-3y}=\frac{x^2-6xy+9y^2}{x-3y}=\frac{(x-3y)^2}{x-3y}=x-3y$$

EXERCISES

1. Reduce fractions:

$$\frac{9}{54};\qquad \frac{x+4}{x^2-16};\qquad \frac{y^3+1}{y+1}.$$

2. Perform operations with fractions:

$$\frac{5}{7}-\frac{2}{7};\qquad \frac{1}{2}+\frac{2}{3};\qquad \frac{3}{5}\times\frac{15}{21};\qquad \frac{4}{7}:\frac{12}{21}.$$

3. Simplify expression: $3\cdot(5x+2)-10x=$

4. Perform factoring operations: $a^3-b^3=$

$b^2-9=$

$(a-b)^2=$

$a^3-125=$

5. Perform operations: $a^4\times a^3=$

$b^8:b^2=$

$b^9:b^{-2}=$

1.2. FUNCTIONAL DEPENDENCE. BASIC FUNCTIONS AND THEIR GRAPHS

Function is the dependence of variable $\boldsymbol{y}$ on the variable $\boldsymbol{x}$ from some set $\boldsymbol{D}$, where each variable value $\boldsymbol{x}$ corresponds a single value of the variable $\boldsymbol{y}$: $\boldsymbol{y=f(x)}$. The equation above is the mainly used representation of a function; it is called the function notation.

The variable $\boldsymbol{x}$ is called independent variable or argument. The variable $\boldsymbol{y}$ is the dependent variable and says that variable $\boldsymbol{y}$ is a function of the variable $\boldsymbol{x}$. All values of the independent variable ***x*** (the set ***D***) are called the domain of definition. The set of all the values taken by the dependent variable ***y*** is called the range of the function

The graph of the function is the set of all points in the coordinate plane, the abscissa of which is equal to the argument values, and the ordinate is the corresponding function values.

Zeros of the function are the values of the argument at which the function vanishes.

The function is called increasing on some interval $\boldsymbol{I}$ if for any $\boldsymbol{x_1, x_2 \in I}$ the inequality $\boldsymbol{x_1 < x_2}$ corresponds to inequality $\boldsymbol{f(x_1) < f(x_2)}$. The function is called decreasing on some interval $\boldsymbol{I}$ if for any $\boldsymbol{x_1, x_2 \in I}$ the inequality $\boldsymbol{x_1 < x_2}$ corresponds to inequality $\boldsymbol{f(x_1) > f(x_2)}$.

The function can be represented in analytical form by formula, in tabular form by means of tables and in graphical form by graph.

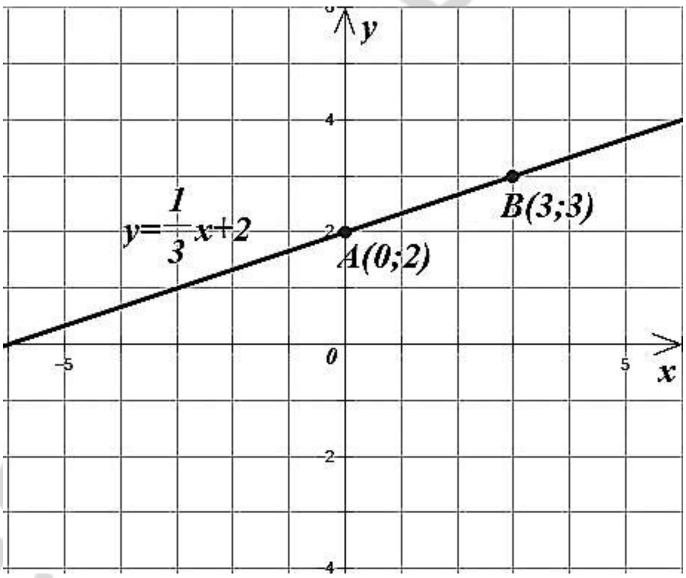
***Example*:**

1) analytical form : $\boldsymbol{y = \frac{1}{3}x + 2}$

2) tabular form:

x	**0**	**3**	**6**	**etc.**
$y = f(x)$	**2**	**3**	**4**	**etc.**

3) graphical form:



1.2.1. LINEAR FUNCTION AND ITS GRAPH

A linear function is a function defined by a formula of the form $\boldsymbol{y = kx + b}$, where $\boldsymbol{x}$ is the argument, $\boldsymbol{k, b \in R}$. The graph of a linear function is a straight line (fig. 1.1).

The coefficient k is called the angular coefficient of the straight line.

Zero of a linear function: $x = -\dfrac{b}{k}$

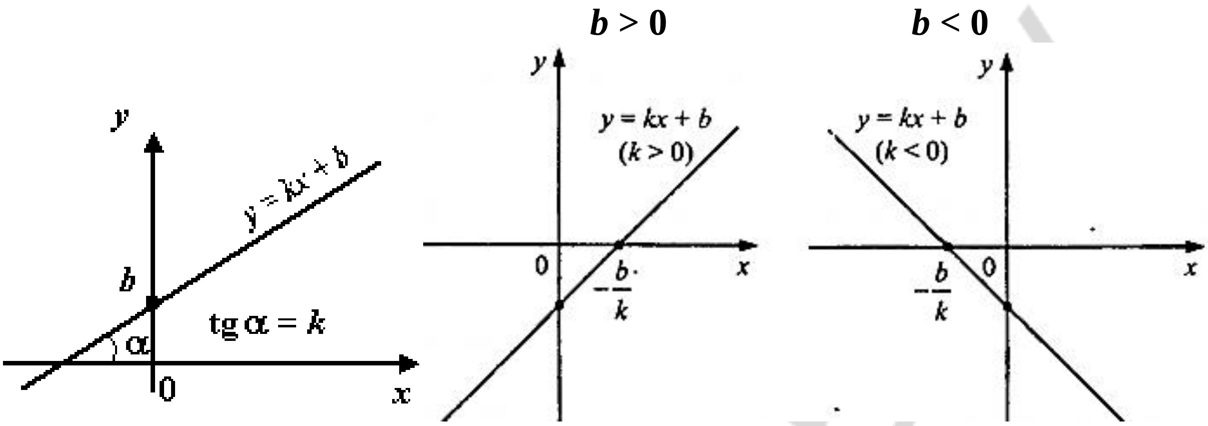


Fig. 1.1. The graph of a linear function

Direct proportionality is a particular case of a linear function (fig. 1.2).

Direct proportionality is a function that can be set by the formula $\boldsymbol{y = kx}$, where $\boldsymbol{x}$ — independent variable, $\boldsymbol{k \neq 0}$. Coefficient $\boldsymbol{k}$ is called the coefficient of direct proportionality.

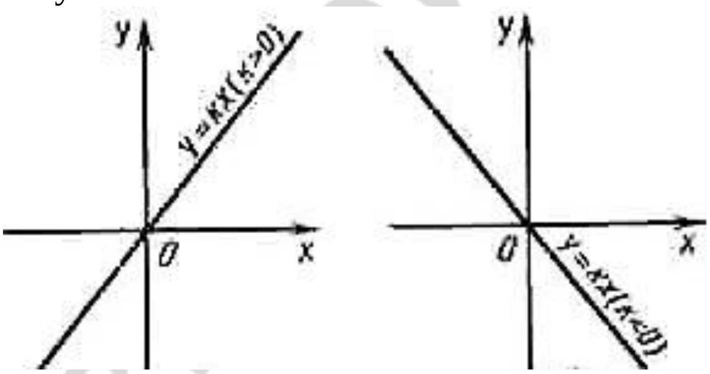


Fig. 1.2. The graph of a direct proportionality function

Exercises

a. Create the graph of the function:

1) $y = 3x$ 2) $y = 2x - 1$ 3) $y = -3x + 2$

b. Find the x-intercept and the y-intercept for the following linear function (zeros of a functions). Find the points of functions intersection:

1) $y = 4x - 6$ and $y = -2x$
2) $y = 2x - 1$ and $y = -4x + 5$
3) $y = 3x - 1$ and $y = -3x + 11$

c. Find a linear function that passes through the origin and forming with the x-axis the following angles:

1) 30º 2) 45º 3) 135º 4) 0º

1.2.2. INVERSE PROPORTIONALITY FUNCTION AND ITS GRAPH

An inverse proportionality function is a function defined by a formula of the form $\boldsymbol{y = \frac{k}{x}}$, where $\boldsymbol{x}$ is the argument, $\boldsymbol{k \in \mathrm{R}, k \neq 0}$.

The domain of this function: $\boldsymbol{x \neq 0}$.

The graph of an inverse proportionality function is a hyperbola (fig. 1.3).

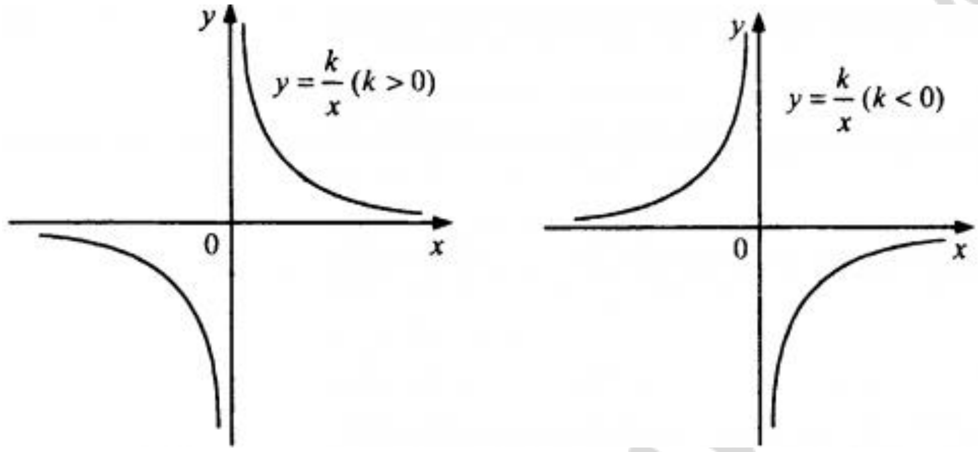


Fig. 1.3. The graph of an inverse proportionality function

There is no zeros of the function $\boldsymbol{y = k/x}$**!**

If $\boldsymbol{k > 0}$**,** the function $\boldsymbol{y = k/x}$ decreases throughout the domain of definition, when $\boldsymbol{k < 0}$, the function $\boldsymbol{y = k/x}$ increases in all the field definitions. For a curve that is the graph of this function, the ***x***-axis and ***y***-axis play the role of asymptotes.

Asymptote — a straight line which is closer to the points of the curve as they remove into infinity.

EXERCISES

a. Create the graph of the function:

1) $y = \frac{3}{x}$ 2) $y = \frac{2}{x}$

1.2.3. QUADRATIC FUNCTION AND ITS GRAPH

A quadratic function is a function defined by a formula of the form:

$$\boldsymbol{y = ax^2 + bx + c,}$$

where $\boldsymbol{x}$ is the argument, $\boldsymbol{a, b, c \in R, a \neq 0}$.

The graph of a quadratic function is a parabola (fig. 2.4).

The vertex of the parabola is the point of intersection of the parabola with its axis of symmetry. The vertex of the parabola $\boldsymbol{y = ax^2 + bx + c}$ has coordinates $\boldsymbol{(-\frac{b}{2a};\ -\frac{b^2 - 4ac}{4a})}$.

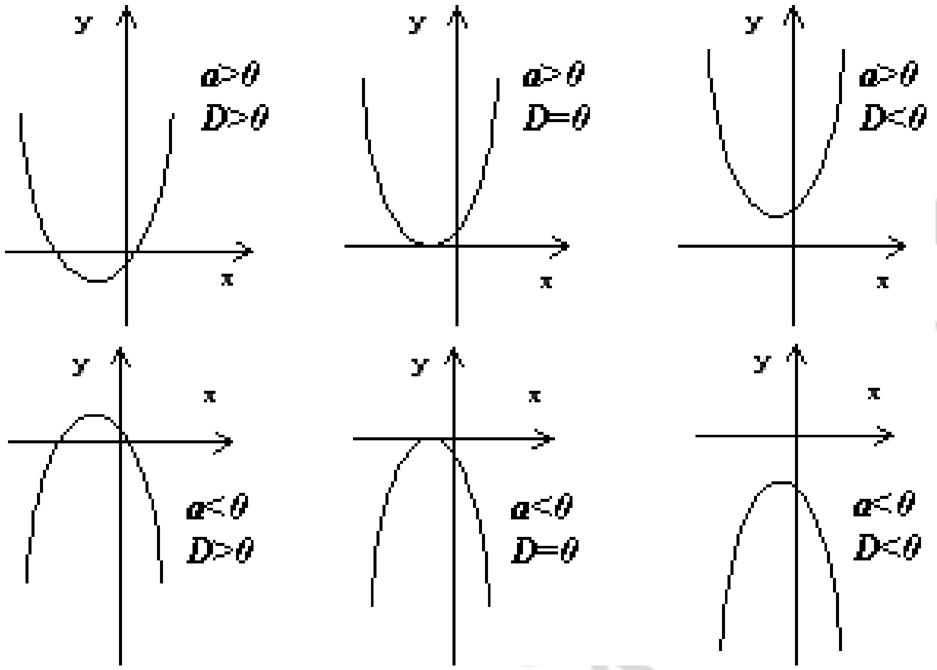


Fig. 1.4. The graph of a quadratic function

Let's consider the function defined by the formula $\boldsymbol{y = ax^2}$ $\mathbf{(a \neq 0)}$ as a particular case of a quadratic function (fig. 1.5).

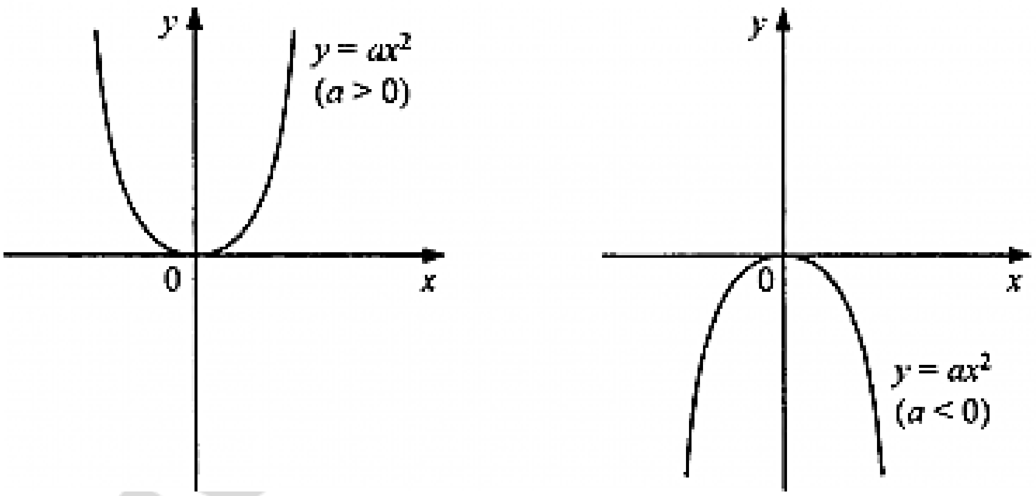


Fig. 1.5. The graph of the function defined by the formula $y = ax^2$ $(a \neq 0)$

The properties of the function $\boldsymbol{y = ax^2}$:

If $\boldsymbol{x = 0, y = 0}$, i. e. the graph of the function passes through the origin.

The function graph is symmetrical about the $\boldsymbol{y}$-axis.

If $\boldsymbol{a > 0}$, the function decreases on the interval $\mathbf{(-\infty; 0]}$ and increases on the interval $\mathbf{[0; +\infty)}$.

If $\boldsymbol{a < 0}$, the function increases on the interval $\mathbf{(-\infty; 0]}$ and decreases on the interval $\mathbf{[0; +\infty)}$.

If $\boldsymbol{a > 0, y_{\min} = 0}$; if $\boldsymbol{a < 0, y_{\max} = 0}$.

1.2.4. QUADRATIC EQUATIONS. QUADRATIC FORMULA

The quadratic equation can be presented in the following form:

$$\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0,$$

where x is a variable, **a**, **b**, **c** are some constants ($\mathbf{a} \neq 0$).

If $\mathbf{a} = 1$ (i. e., the equation of the form $x^2 + \mathbf{b}x + \mathbf{c} = 0$), the quadratic equation is called monic quadratic equation.

Quadratic Formula: If $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$, then $x = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$.

In the general form of a quadratic equation $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$. The expression $\mathbf{D} = \mathbf{b}^2 - 4\mathbf{ac}$ called the discriminant of the quadratic equation $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$.

If $\mathbf{D} < 0$, the quadratic equation has no roots.

When $\mathbf{D} = 0$, the quadratic equation has two of the same root: $x = -\dfrac{\mathbf{b}}{2\mathbf{a}}$.

If $\mathbf{D} > 0$, the quadratic equation has two roots $x_{1,2} = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{D}}}{2\mathbf{a}}$.

It is useful to remember the following factoring formula for practical use:

$$\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = \mathbf{a}\,(x - x_1) \cdot (x - x_2).$$

Viete Theorem: If x_1 and x_2 are the roots of the monic quadratic equation $x^2 + \mathbf{p}x + \mathbf{q} = 0$, then $x_1 + x_2 = -\mathbf{p}$; $x_1 \cdot x_2 = \mathbf{q}$.

***Example*:**

Find the roots of the quadratic equation $x^2 - 2x - 3 = 0$.

***Solution*:** $D = (-2)^2 - 4\cdot 1\cdot(-3) = 4 + 12 = 16 = 4^2$ $\quad x_{1,2} = \dfrac{-(-2) \pm \sqrt{4^2}}{2\cdot 1}$

$x_1 = \dfrac{2-4}{2} = -1$; $\quad x_2 = \dfrac{2+4}{2} = 3$.

Answer: –1; 3.

EXERCISES

a. Create the graph of the function:

1) $y = 3x^2 + 2x - 1$; 2) $y = -x^2 + 2$; 3) $y = -4x^2$.

b. Find the roots of the quadratic equation:

1) $x^2 + x - 20 = 0$; 2) $x^2 - 8x - 9 = 0$; 3) $16x^2 - 40x + 25 = 0$.

4) $x^2 - 6x + -6 = 0$; 5) $x^2 + \dfrac{1}{3}x - \dfrac{2}{3} = 0$; 6) $2x^2 - 5x - 7 = 0$.

1.2.5. CUBIC FUNCTION AND ITS GRAPH

Cubic function in mathematics is a numerical function of the following form $\quad f(x) = \mathbf{a}x^3 + \mathbf{b}x^2 + \mathbf{c}x + \mathbf{b},\ x \in \mathbf{R}, \quad$ where $\mathbf{a} \neq 0$.

Generally speaking, a cubic function is a polynomial of the third degree. The graph of a full cubic function is the following (fig. 1.6).

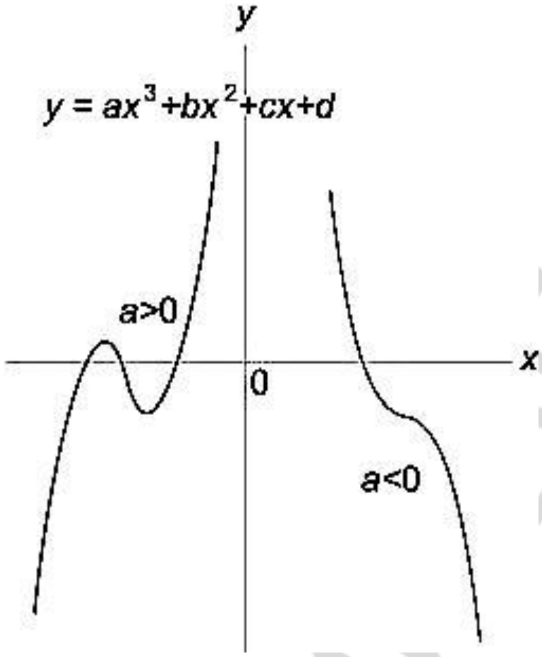


Fig. 1.6. The graph of a full cubic function

Let's consider the function defined by the formula $\boldsymbol{y = ax^3\ (a \neq 0)}$. In this case the graph of the cubic function is a cubic parabola (fig. 1.7).

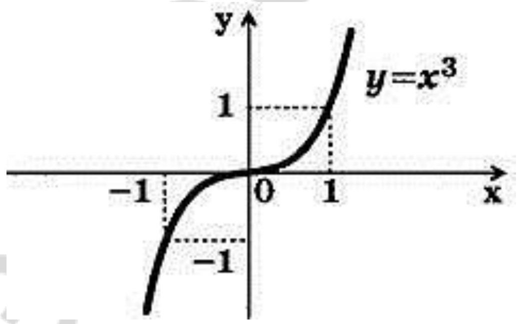


Fig. 1.7. The graph of the function defined by the formula $y = ax^3$ $(a \neq 0)$

Cubic parabola — a plane algebraic curve of the third order. Its canonical equation in rectangular Cartesian coordinates has the form $\boldsymbol{y = ax^3}$, where $\boldsymbol{a \neq 0}$.

The cubic parabola has a center of symmetry at the origin, this point is the inflection point of the curve. The ***x***-axis is the tangent to the cubic parabola at that point.

For **a > 0** cubic parabola is located in the first and third quarters of the coordinate, it is an increasing function.

For **a < 0** the curve runs in the second and fourth quarters and decreases.

Cubic parabola exists at least one the x-intercept but no more than three ***x***-intercepts. Axis from 1 to 3 times. Sit means, that the cubic equation $\boldsymbol{ax^3 + bx^2 + cx + b = 0}$ has at least one up to three roots.

EXERCISES

a. Create the graph of the function:
1) $y = 2x^3$; 2) $y = 3x^3$.

1.2.6. THE EXPONENTIAL FUNCTION AND ITS GRAPH

An exponential function is a function defined by a formula of the form

$$\boldsymbol{y = a^x},$$

where $\boldsymbol{a > 0, a \neq 1}$ (fig. 1.8).

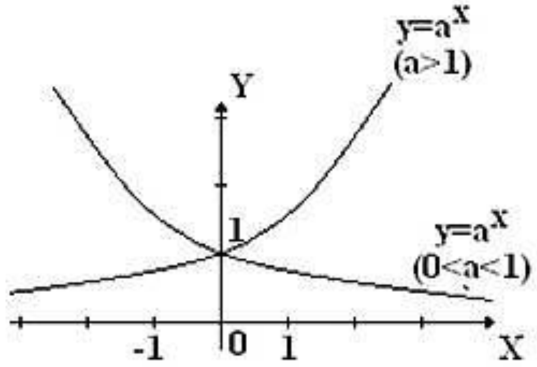


Fig. 1.8. The graph of an exponential function

In contexts where the base **a** is not specified, especially in more theoretical contexts, the term **exponential function** is almost always understood to mean the **natural exponential function** $\boldsymbol{y = e^x}$ (fig. 1.9).

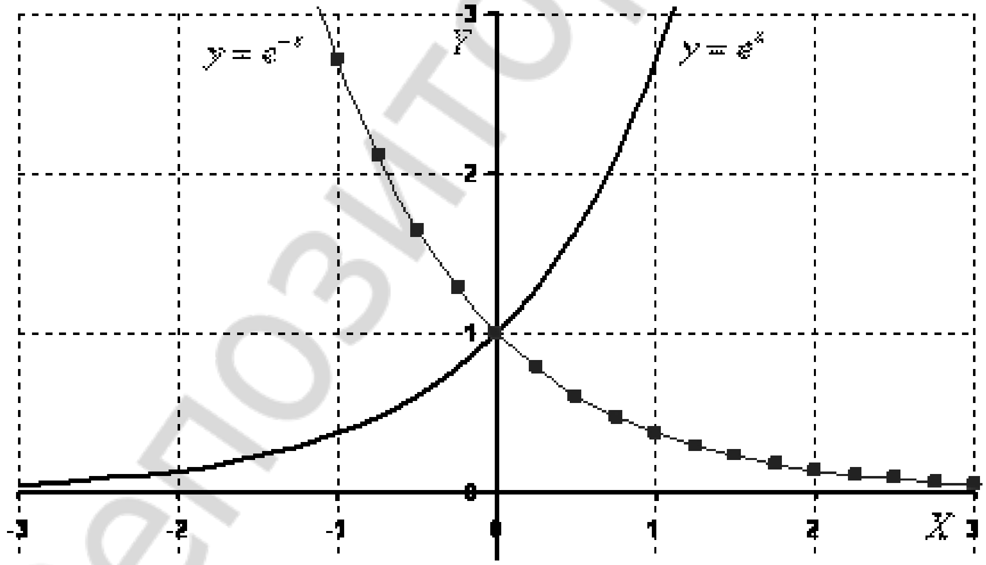


Fig. 1.9. The graph of the natural exponential function $y = e^x$

Let's consider the function defined by the formula $\boldsymbol{y = e^x}$ (where **e** is Euler's or irrational number, **e = 2.71828**) as a particular case of an exponential function.

The exponential function is used to model a relationship in which a constant change in the independent variable gives the same proportional change (i. e. percentage increase or decrease) in the dependent variable. The function is often written as **exp(*x*)**. The exponential function is widely used in physics, biology and mathematics.

***Example*:** Radioactivity is one very frequently given example of exponential decay. The law describes the statistical behavior of a large number of nuclides, rather than individual atoms.

$$\boldsymbol{N(t) = N^0\ e^{-\lambda t}}.$$

Here $\boldsymbol{N(t)}$ is the quantity at time $\boldsymbol{t}$, and $\boldsymbol{N_0 = N(0)}$ is the initial quantity, i. e. the quantity at time $\boldsymbol{t = 0}$, and **λ** (lambda) is a positive rate called the exponential decay constant.

1.2.7. LOGARITHM. COMMON AND NATURAL LOGARITHMS. THE PROPERTIES OF LOGARITHMS. LOGARITHMIC FUNCTION AND ITS GRAPH

If at an exponential function $\boldsymbol{y = a^x}$ change the places of $\boldsymbol{x}$ and $\boldsymbol{y}$ we will have a function $\boldsymbol{x = a^y}$. So $\boldsymbol{y}$ is the power of base $\boldsymbol{a}$ to calculate value $\boldsymbol{x}$, it means that $\boldsymbol{y = \log_a x}$, *y is the logarithm of* ***x*** *on base* ***a.***

That is, the logarithm of a number $\boldsymbol{x}$ to the base $\boldsymbol{a}$ is that number $\boldsymbol{y}$ which, as the exponent of $\boldsymbol{a}$, gives back the number $\boldsymbol{x}$.

For common logarithms, the base is 10, so if $\boldsymbol{x = 10^y}$, then $\boldsymbol{y = \lg x}$ or $\boldsymbol{\log x}$. The subscript **10** on $\boldsymbol{\log_{10}}$ is usually omitted when dealing with common logs.

Another important base is the exponential base **e = 2.71828**..., where **e** is natural number (or Euler's or irrational number). Such logarithms are called natural logarithms and are written **ln**.

Thus, if $\boldsymbol{x = e^y}$, then $\boldsymbol{y = \ln x}$.

For any number $\boldsymbol{y}$, the two types of logarithm are related by the equality:

$\boldsymbol{\ln x = 2.3026 \log x}$.

Some simple rules for logarithms:

1. $\boldsymbol{\log (xz) = \log x + \log z}$,

which is true because if $\boldsymbol{x = 10^n}$ and $\boldsymbol{z = 10^m}$, then $\boldsymbol{xz = 10^{n+m}}$. From the definition of logarithm, $\boldsymbol{\log x = n}$, $\boldsymbol{\log z = m}$, and $\boldsymbol{\log (xz) = n + m}$; hence, $\boldsymbol{\log(xz) = n + m = \log x + \log z}$.

In a similar way, we can show that.

2. $\boldsymbol{\log \frac{x}{z} = \log x - \log z}$.

3. $\boldsymbol{\log x^n = n \log x}$.

These three rules apply to any kind of logarithm. The function that assigns to $\boldsymbol{y}$ its logarithm is called logarithm function or logarithmic function (or just logarithm) $\boldsymbol{y = \log_a x}$, ***if*** $\boldsymbol{a > 0}$, $\boldsymbol{a \neq 1}$, $\boldsymbol{x > 0}$.

The graph of a classical logarithmic function $\boldsymbol{y = \log_a x}$ will be the following (fig. 1.10).

Let's consider the graph of a natural logarithmic function $\boldsymbol{y = \ln x}$ as a particular case of a logarithmic function (fig. 1.11).

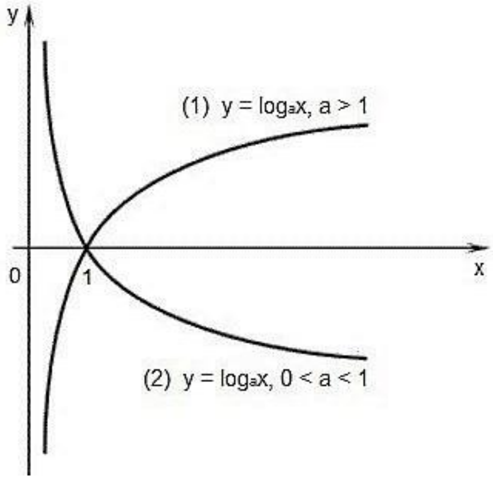


Fig. 1.10. The graph of a classical logarithmic function

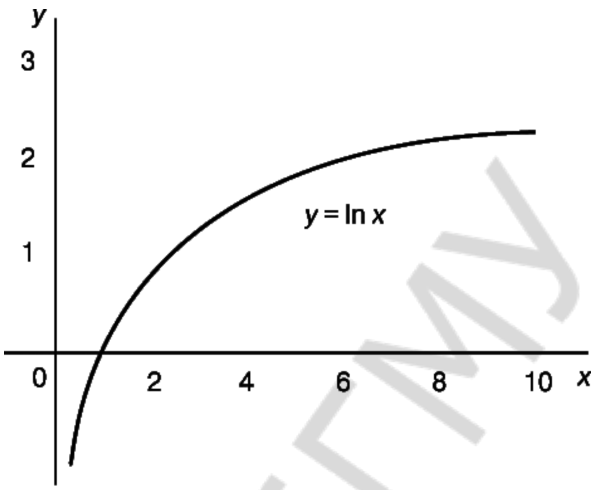


Fig. 1.11. The graph of a natural logarithmic function

EXERCISES

a. Find the logarithms:

1) $\log_6 2 + \log_6 18$; 2) lg4 + lg25; 3) lg3000 – lg3; 4) ln e^5.

b. Find *x*:

1) $3 = \log_2(15 - x)$; 2) $5 = \lg(100^{-x})$; 3) $6 = \ln(e^{2x})$.

1.2.8. TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS. PROPERTIES OF TRIGONOMETRIC FUNCTIONS

Trigonometric functions — math functions of an angle. They are certainly important when studying geometry, and in the study of periodic processes. Typically trigonometric functions are defined as the relationship of sides of a right triangle or the length of certain segments in the unit circle.

An angle can be set in radians (1 rad) and degrees (1º). One radian is equal to 180/π degrees. Thus, to convert from radians to degrees, multiply by 180/π.

$$1\text{ rad} = 1\cdot\frac{180^\circ}{\pi} \approx 57.2958^\circ \qquad 1^\circ = 1\cdot\frac{\pi}{180^\circ} \approx 0{,}0175\text{ rad.}$$

Correspondence between the main trigonometric functions presented in table 1.2.

Table 1.2

Trigonometric functions

Functions	Correspondence
sin	$\sin x = \cos\left(\frac{\pi}{2} - x\right)$
cos	$\cos x = \sin\left(\frac{\pi}{2} - x\right)$
tg *or* tan	$\operatorname{tg} x = \frac{\sin x}{\cos x} = \operatorname{ctg}\left(\frac{\pi}{2} - x\right) = \frac{1}{\operatorname{ctg} x}$
ctg *or* cot	$\operatorname{ctg} x = \frac{\cos x}{\sin x} = \operatorname{tg}\left(\frac{\pi}{2} - x\right) = \frac{1}{\operatorname{tg} x}$

a. Function ***y* = sin *x*** and its graph (fig. 1.12).

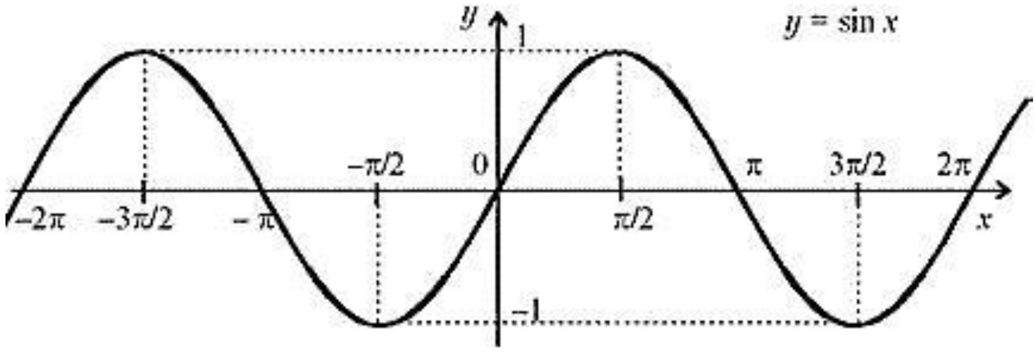


Fig. 1.12. The graph of a function $y = \sin x$

b. Function ***y* = cos *x*** and its graph (fig. 1.13).

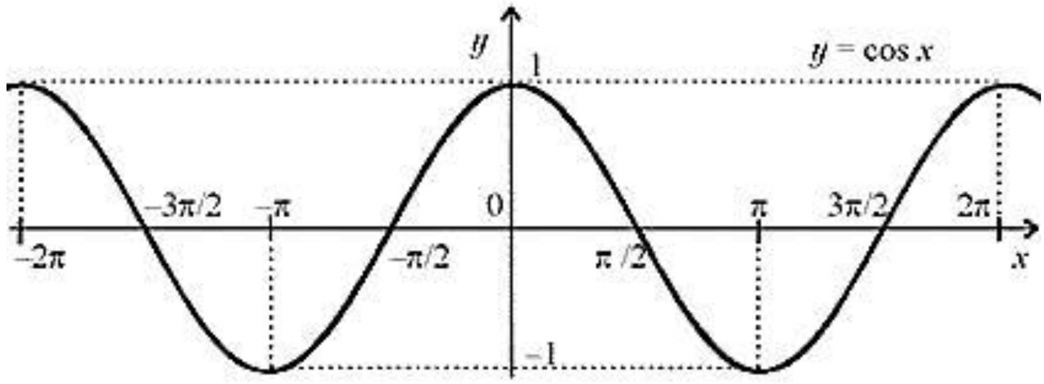


Fig. 1.13. The graph of a function $y = \cos x$

Trigonometric functions ***y* = sin *x*** and ***y* = cos *x*** are periodical functions with period $T = 2\pi$.

c. Function ***y* = tg *x*** (***y* = tan *x***) and its graph (fig. 1.14).

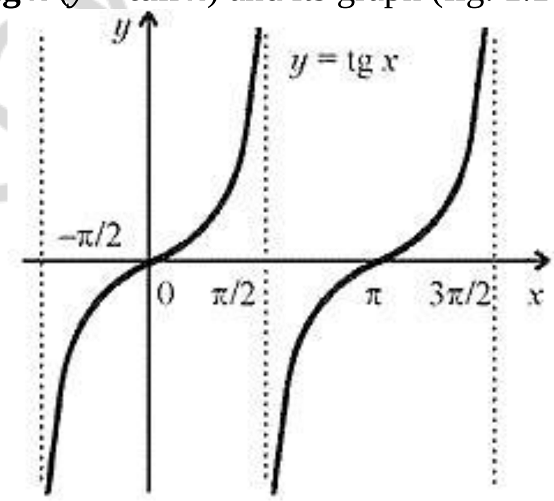


Fig. 1.14. The graph of a function $y = \text{tg}\, x$ ($y = \tan x$)

Trigonometric function ***y* = tg *x*** is periodical function with period $\boldsymbol{T = \pi}$.

Table 1.3

Trigonometric functions of the main angles

α (rad)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
α (deg)	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin α	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos α	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	–1
tg α	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0
ctg α	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	–

We take an arbitrary right triangle that contains the angle **α** to define trigonometric functions of the angle **α**. We will set the sides of this triangle as:

Hypotenuse is a side opposite right angle (longest side in the triangle), the ***c*** side in this case.

The opposite leg is the side that lies opposite the angle **α**. For example, side ***a*** is opposite to angle **α**.

The adjacent leg is the side which is a party angle. For example, leg ***b*** is adjacent to angle **α** (fig. 1.15).

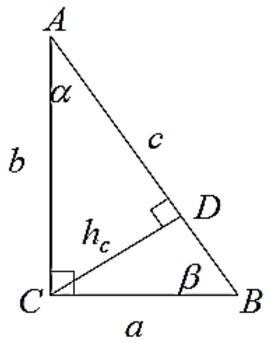


Fig. 1.15. An arbitrary right triangle

The ***sine*** of angle **α** is the ratio of the length of the opposite side ***a*** to the length of the hypotenuse ***c***:

$$\sin\alpha = \frac{a}{c}.$$

This attitude does not depend on the choice of triangle {ABC} containing the angle **α**, since all such triangles are similar.

The ***cosine*** of angle **α** is the ratio of the length of the adjacent side ***b*** to the length of the hypotenuse ***c***: $\cos\alpha = \frac{b}{c}$.

The ***sine*** of one acute angle in the triangle equals the cosine of the second:

$$\sin\beta = \frac{b}{c} = \cos\alpha = \frac{b}{c}.$$

The ***tangent*** of angle **α** is the ratio of the length of the opposite side ***a*** to the length of the adjacent side ***b***: $\mathrm{tg}\,\alpha = \frac{a}{b}$.

This is easy to see by studying a right triangle and applying the Pythagorean theorem that in symbolic form the Pythagorean identity can be written as: $\sin^2\alpha + \cos^2\alpha = 1.$

1.2.9. Main trigonometric formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

$$\mathrm{tg}(\alpha \pm \beta) = \frac{\mathrm{tg}\,\alpha \pm \mathrm{tg}\,\beta}{1 \pm \mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\mathrm{tg}\,\alpha}{1+\mathrm{tg}^2\alpha} = \frac{2\mathrm{ctg}\,\alpha}{1+\mathrm{ctg}^2\alpha} = \frac{2}{\mathrm{tg}\,\alpha + \mathrm{ctg}^2\alpha}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \frac{1-\mathrm{tg}^2\alpha}{1+\mathrm{tg}^2\alpha} = \frac{\mathrm{ctg}^2\alpha - 1}{\mathrm{ctg}^2\alpha + 1}$$

$$\tan 2\alpha = \frac{2\mathrm{tg}\,\alpha}{1-\mathrm{tg}^2\alpha} = \frac{2\mathrm{ctg}\,\alpha}{\mathrm{ctg}^2\alpha - 1} = \frac{2}{\mathrm{ctg}\,\alpha - \mathrm{tg}\,\alpha}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}$$

$$\sin\alpha\cos\beta = \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2}$$

$$\mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{\cos(\alpha-\beta) + \cos(\alpha+\beta)}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} = \frac{\mathrm{tg}^2\alpha}{1+\mathrm{tg}^2\alpha}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} = \frac{\mathrm{ctg}^2\alpha}{1+\mathrm{ctg}^2\alpha}$$

$$\mathrm{tg}^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} = \frac{\sin^2\alpha}{1-\sin^2\alpha}$$

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$

Exercises

a. Simplify the expression:

1) sin 20° cos 40° + cos 20° sin 40°; 2) sin 45° cos 15° + cos 45° sin 15°;
3) cos 12° cos 18° – sin 12° sin 18°; 4) cos 98° cos 8° – sin 98° sin 8°;

5) $\dfrac{\text{tg }22^\circ + \text{tg }23^\circ}{1 - \text{tg }22^\circ \text{tg }23^\circ}$; 6) $\dfrac{\text{tg }45^\circ - \text{tg }15^\circ}{1 + \text{tg }15^\circ \text{tg }45^\circ}$.

b. Create the graph of the function:

1) y = cos2x; 2) y = sin3x.

c. Calculate:

1) cos120º; 2) cos135º; 3) sin75º; 4) tg1105º.

1.3. VECTORS

A **geometric vector** (or simply a **vector**) is a geometric object that has length or magnitude and direction. It can be added to other vectors according to the rules of vector algebra. A vector is frequently represented in mathematics and physics by a line segment with a definite direction, or graphically as an arrow, connecting an *initial point A* with a *terminal point B*, and denoted by $\overrightarrow{AB}$.

The vector can also be denoted as $\overrightarrow{AB}$ if it represents a directed distance or displacement from a point A to a point B (fig. 1.16).

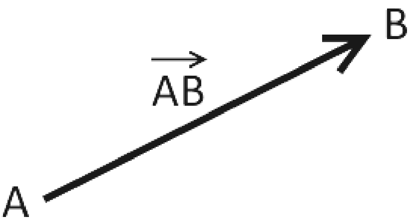


Fig. 1.16. The vector $\overrightarrow{AB}$

Latin word *vector* means “carrier”. A vector is what is needed to “carry” the point A to the point B. The magnitude of the vector is the distance between the two points and the direction refers to the direction of displacement from A to B. Mathematical operations on real numbers such as addition, subtraction and multiplication have close analogues for vectors. Vectors play an important role in physics. Velocity and acceleration of a moving object as well as forces acting on it are described by vectors. Although most of other physical quantities do not represent distances (except, for example, position and displacement), their magnitude and direction can be still represented by the length and direction of an arrow. The mathematical representation of a physical vector depends on the coordinate system used to describe it.

1.3.1. Vector addition

It is possible to use different rules (methods) for the geometric construction of the vector addition $\vec{\mathbf{a}} + \vec{\mathbf{b}}$, but they all give the same result.

a. Triangle rule.

Both vectors are transported parallel to themselves so that the beginning of one of them coincided with the end of another. Then the vector sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is defined by a third party resulting triangle, and its beginning coincides with the beginning of the first vector $\vec{\mathbf{a}}$ and the end of the second vector $\vec{\mathbf{b}}$ (fig. 1.17).

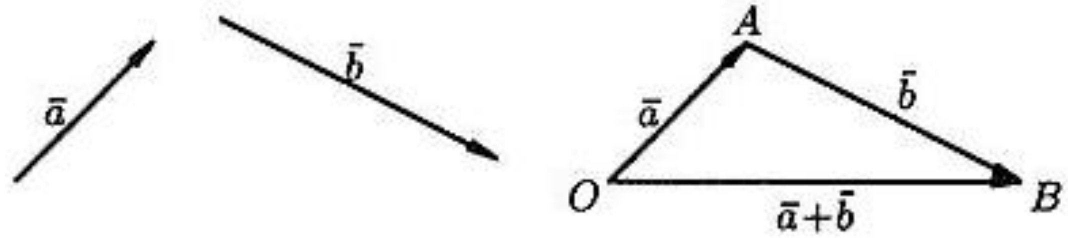


Fig. 1.17. The vector sum $\vec{\mathbf{a}} + \vec{\mathbf{b}}$

This rule is directly and naturally generalized to add any number of vectors, moving in the rule of the polygon.

b. Polygon rule.

The beginning of the second vector coincides with the end of the first, the beginning of the third — with the end of the second, and so on. The sum of ***n*** vectors is a vector with the beginning coinciding with the beginning of the first vector and the end coinciding with the end of the ***n***-th vector (fig. 1.18).

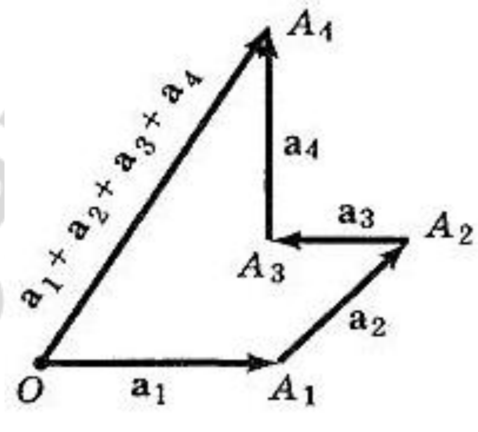


Fig. 1.18. The sum of ***n*** vectors

c. Parallelogram rule.

Both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are transported parallel to themselves so that their beginning matches. Then the vector sum is given by diagonal built on a parallelogram of them coming from their common base point (fig 1.19).

The rule of the parallelogram is especially useful when there is a need to represent the vector of amounts immediately applied to the same point, which is applied to both terms — that is, to depict all three vectors having a common origin.

$$a - b = (a_1 - b_1)e_1 + (a_2 - b_2)e_2 + (a_3 - b_3)e_3.$$

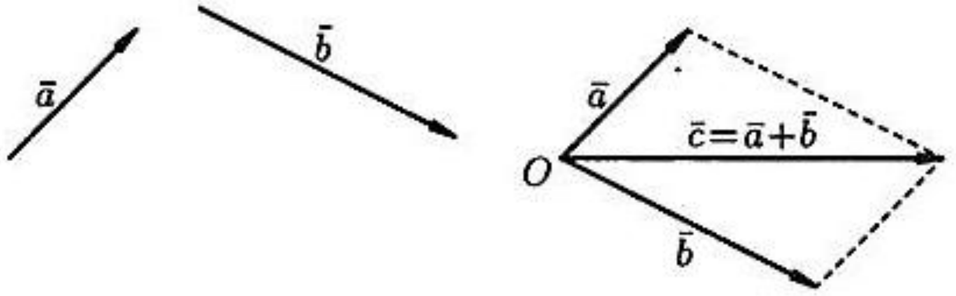


Fig. 1.19. Parallelogram rule

1.3.2. VECTOR SUBTRACTION

Subtraction of two vectors can be geometrically defined as follows: to subtract $\vec{\mathbf{b}}$ from, $\vec{\mathbf{a}}$ place the tails of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ at the same point, and then draw an arrow from the head of $\vec{\mathbf{b}}$ to the head of **a**. This new arrow represents the vector $\vec{\mathbf{a}} - \vec{\mathbf{b}}$, as illustrated below on fig. 1.20.

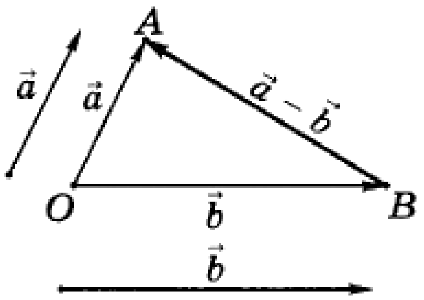


Fig. 1.20. Subtraction of two vectors

Subtraction of two vectors may also be performed by adding the opposite of the second vector to the first vector, that is $\vec{\mathbf{a}} - \vec{\mathbf{b}} = \vec{\mathbf{a}} + (-\vec{\mathbf{b}})$.

1.3.3. VECTOR MULTIPLICATION (SCALAR MULTIPLICATION)

a. Multiplication by a number.

Multiplication of a vector $\vec{\mathbf{a}}$ by a number ***n*** > **0** gives a collinear vector with length n times greater.

Multiplication of a vector $\vec{\mathbf{a}}$ by a number ***n*** < **0** gives the opposite directional vector with a length of n times more (fig. 1.21).

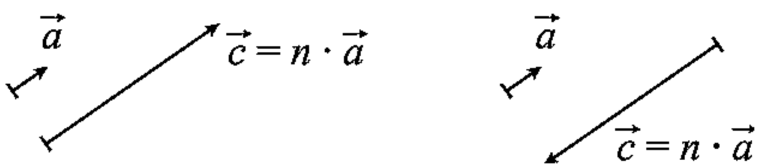


Fig. 1.21. Multiplication of a vector $\vec{\mathbf{a}}$ by a number ***n*** > **0**

b. The dot product.

For geometric vectors the dot product is defined via their geometric features and is entered as follows:

$$\vec{\mathbf{a}}\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\mathbf{cos}(\vec{\mathbf{a}}, \vec{\mathbf{b}}).$$

Here to calculate the cosine takes the angle between the vectors, which is defined as the angle formed by the vectors, if you put them to one point (to combine them).

1.3.4. VECTOR DECOMPOSITION

An arbitrary vector $\vec{\mathbf{c}}$ can be represented as a sum: $\vec{\mathbf{c}} = m\vec{\mathbf{a}} + n\vec{\mathbf{b}}$, where *m* and *n* are arbitrary numbers, and the triple of vectors $\vec{\mathbf{c}}$, **a** and **b** are coplanar (fig. 1.22). It is a decomposition of the vector **c** on $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ components. If the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not collinear, the submitted decomposition is the only possible.

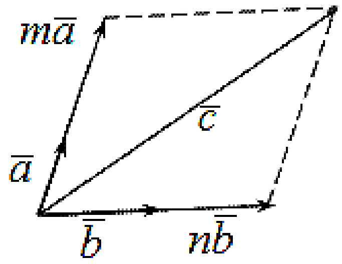


Fig. 1.22. Vector decomposition

1.3.5. PROJECTION OF VECTOR ON A COORDINATE AXIS

The projection of the vector $\overrightarrow{\mathbf{AB}}$ on the axis $\boldsymbol{l}$ is a number equal to the size of the segment $\mathbf{A_1B_1}$ to the axis $\boldsymbol{l}$, where the points $\mathbf{A_1}$ and $\mathbf{B_1}$ are the projections of points **A** and **B** on the axis (fig. 1.23).

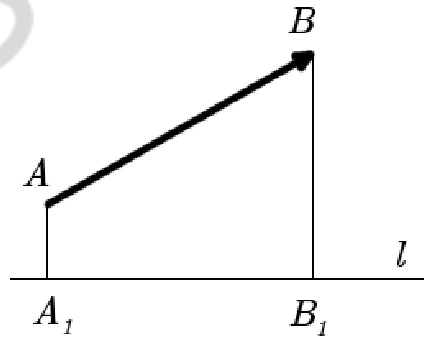


Fig. 1.23. The projection of the vector $\overrightarrow{\mathbf{AB}}$ on the axis $\boldsymbol{l}$

The projection of the vector on the coordinate axis is a scalar value. The sign of the projection depends on the direction of the vector relative to the coordinate axes.

Projection may be positive or negative. The projection of the vector $\overrightarrow{\mathbf{AB}}$ on an some axis is called positive if the projection direction coincides with the direction of the axis from the beginning of the projection up to the end of it.

Let's imagine $\mathbf{C}_x$ as the projection of the vector $\vec{\mathbf{c}}$ on the axis $\boldsymbol{x}$ (fig. 1.23).

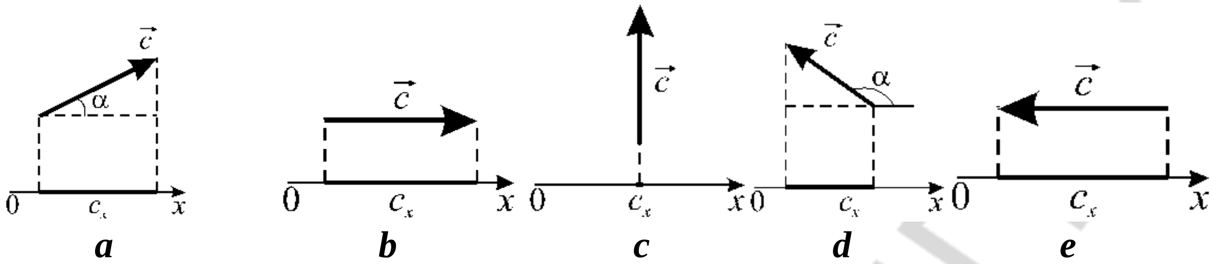


Fig. 1.24. The projections of the vector $\vec{\mathbf{c}}$ on the axis $\boldsymbol{x}$

a. if $\mathbf{0° < \alpha < 90°}$, $\mathbf{c}_x = \mathbf{c \cdot \cos \alpha}$, $\mathbf{0 < \cos \alpha < 1}$, $\mathbf{c}_x > \mathbf{0}$.

b. if $\mathbf{\alpha = 0°}$, $\vec{\mathbf{c}} \uparrow\uparrow \mathbf{O}\boldsymbol{X}$, $\mathbf{c}_x = \mathbf{c \cdot \cos 0°}$, $\mathbf{c}_x = \mathbf{+c}$, projection is positive.

c. if $\mathbf{\alpha = 90°}$, $\vec{\mathbf{c}} \perp \mathbf{O}\boldsymbol{X}$, $\mathbf{c}_x = \mathbf{c \cdot \cos 90°}$, $\mathbf{c}_x = \mathbf{0}$.

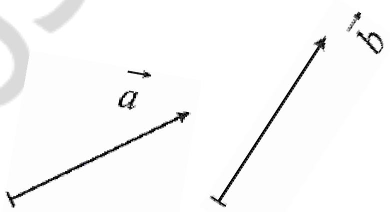
d. if $\mathbf{180° > \alpha > 90°}$, $\mathbf{-1 < \cos \alpha < 0}$, $\mathbf{c}_x < \mathbf{0}$.

e. if $\mathbf{\alpha = 180°}$, $\vec{\mathbf{c}} \uparrow\downarrow \mathbf{O}\boldsymbol{X}$, $\mathbf{c}_x = \mathbf{c \cdot \cos 180°}$, $\mathbf{c}_x = \mathbf{-c}$, projection is negative.

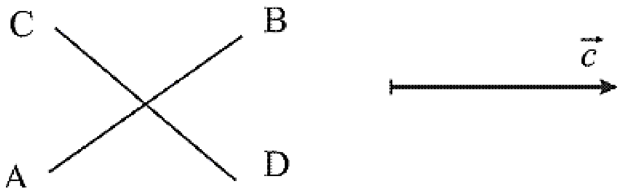
Exercises

a. Produce an addition of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$:

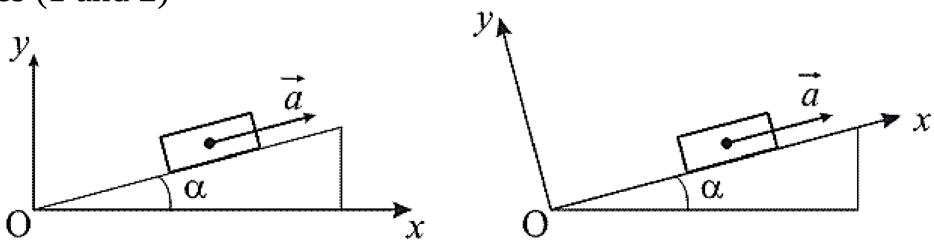
b. Produce a subtraction of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:



c. Produce a decomposition of the vector $\vec{\mathbf{c}}$ on $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ components along the directions AB and CD



d. Produce a projection of vector $\vec{\mathbf{a}}$ on the coordinate axises $\boldsymbol{x}$ and $\boldsymbol{y}$ for both cases (1 and 2)



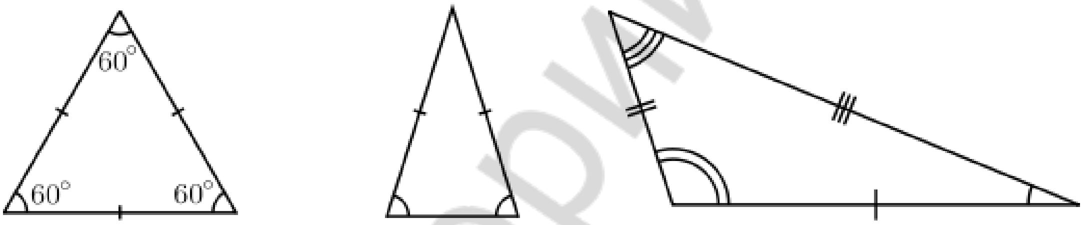
1 **2**

1.4. ELEMENTARY GEOMETRY FIGURES AND FORMULAS

Triangle. Types of triangles.

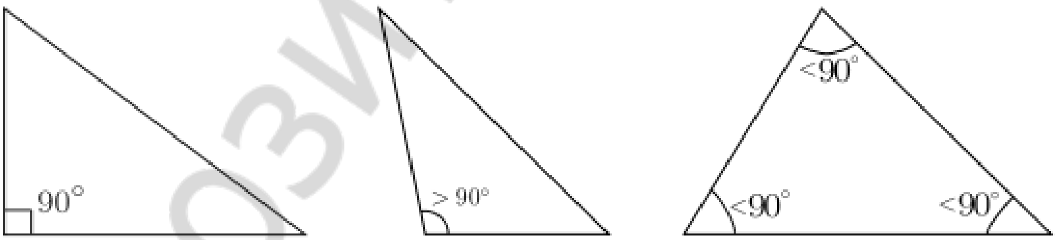
Triangles can be classified according to the lengths of their sides: Equilateral (*a*), Isosceles (*b*) and Scalene (*c*) Triangles (fig. 1.25).

Triangles can also be classified according to their internal angles: Right (*a*), Obtuse (*b*) and Acute (*c*) Triangles (fig. 1.26).



a ***b*** ***c***

Fig. 1.25. Triangles classification according to the length of their sides:
a — Equilateral; *b* — Isosceles; *c* — Scalene



a ***b*** ***c***

Fig. 1.26. Triangles classification according to the internal angles:
a — Right; *b* — Obtuse; *c* — Acute

The Pythagorean theorem.

In any right triangles, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the two other sides. If the hypotenuse has length $\boldsymbol{c}$, and the legs have lengths $\boldsymbol{a}$ and $\boldsymbol{b}$ (fig. 1.27) then the theorem states that $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$.

The area $\boldsymbol{S}$ of the triangle (fig. 1.28): $\boldsymbol{S} = \dfrac{1}{2}\boldsymbol{bh}$.

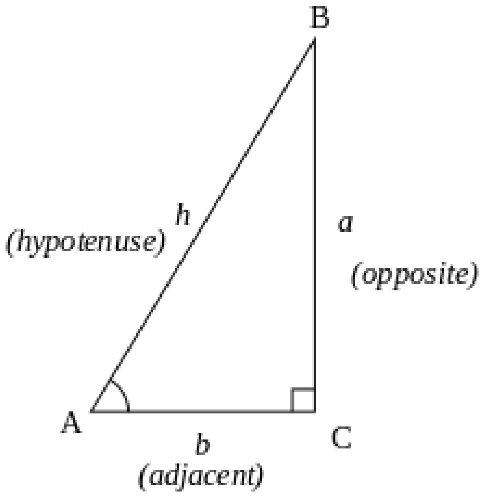


Fig. 1.27. The Pythagorean theorem for right triangles

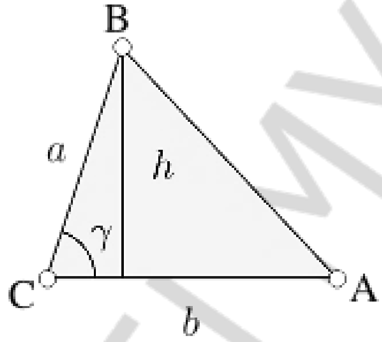


Fig. 1.28. The area ***S*** of the triangle

The area ***S*** of the parallelogram (fig. 1.29): $\boldsymbol{S = ah = ab \sin \alpha}$.

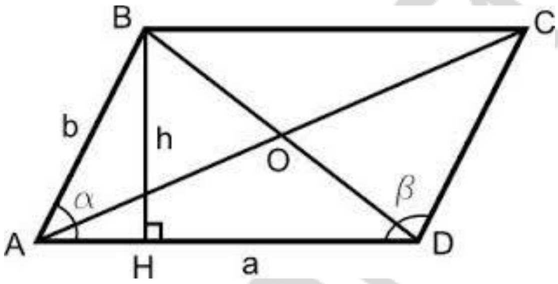


Fig. 1.29. The area ***S*** of the parallelogram

The length ***L*** of a circle (fig. 1.30): $\boldsymbol{L = 2\pi R = \pi D}$.
The surface ***S*** area of a circle (fig. 1.30): $\boldsymbol{S = \pi R^2}$.

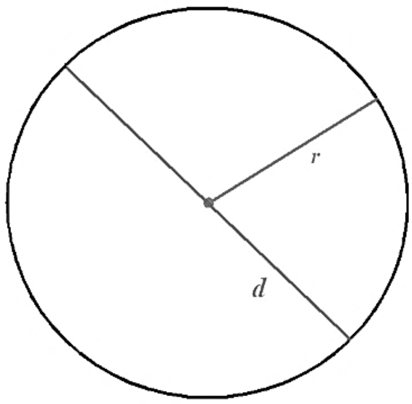


Fig. 1.30. The surface area ***S*** of a circle

A **rectangle** is any quadrilateral with four right angles. It can also be defined as a parallelogram containing a right angle (fig. 1.31).

If a rectangle has length ***a*** and width ***b***, its area: $\boldsymbol{S = ab}$, and its perimeter $\boldsymbol{P = 2(a + b)}$.

A rectangle with four sides of equal length is a square (fig. 1.32).

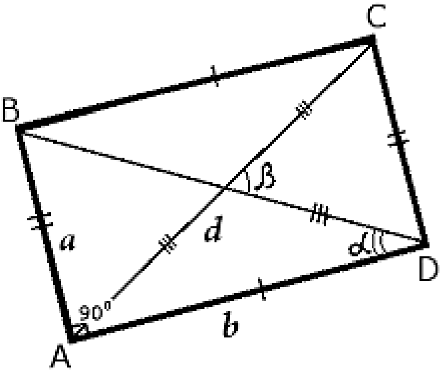


Fig. 1.31. A rectangle

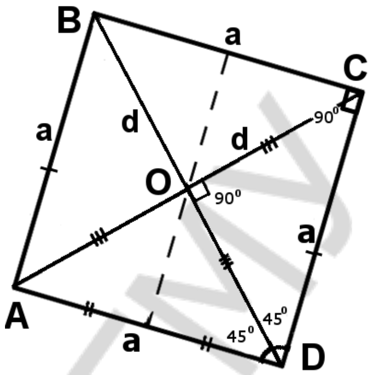


Fig. 1.32. A square

The perimeter $\boldsymbol{P}$ of a square whose four sides have length $\boldsymbol{a}$ is: $\boldsymbol{P = 4a}$ and the area $\boldsymbol{S}$ is: $\boldsymbol{S = a^2}$ or it can be calculated using the diagonal $\boldsymbol{d}$: $\boldsymbol{S} = \dfrac{\mathbf{d}^2}{\mathbf{2}}$.

1.5. LIMIT OF A FUNCTION

The number $\boldsymbol{L}$ is called the limit of a function $\boldsymbol{f(x)}$ at a point $\boldsymbol{p}$ if for any sequence $\{x_n\} \subset \mathrm{D}[f]$ that converges to a point $\boldsymbol{p}$, the corresponding sequence of function values $\{f(x_n)\}$ converges to $\boldsymbol{L}$.

$$\lim_{x \to p} f(x) = L.$$

It means generally that $\boldsymbol{f(x)}$ can be made as close as desired to $\boldsymbol{L}$ by making $\boldsymbol{x}$ close enough, but not equal, to $\boldsymbol{p}$.

1.5.1. LIMITS OF SPECIAL INTEREST

Trigonometric functions

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$ — *First wonderful limit*

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Exponential functions

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e = 2.7178282$$ — *Second wonderful limit*

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

c. Logarithmic functions $$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

***Examples*:**

1. We will set the function $y = 3x^2 + 2x - 5$ and the argument x of the function tends to 2. If the argument of the function takes on the values 2.1; 2.01; 2.001 etc. → 2, the function gets the values 12.43; 11.14; 11.01 etc. → 11. That is, the sequence of function values has a limit, equals to 11.

We can write down the following expression: $\lim\limits_{x\to 2}(3x^2 + 2x - 5) = 11$.

Thus we can simply substitute the value of the argument limit directly to the to the function expression in order to calculate the value of the function limit.

2. Find the limit of the function $y = \dfrac{x^2 - 4}{x - 2}$ if $x \to 2$.

It is not possible to make a direct substitution the value of the argument limit directly to the to the function expression because we will obtain no sense result such as $\dfrac{0}{0}$. We may use factoring formula $(a^2 - b^2) = (a - b)(a + b)$

$$\lim_{x\to 2}\frac{x^2 - 4}{x - 2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x - 2} = \lim_{x\to 2}(x + 2) = 2 + 2 = 4.$$

EXERCISES

a. calculate the limits of functions:

1. $\lim\limits_{x\to 2}(x^3 - 4x + 1)$; 2. $\lim\limits_{x\to -2}\dfrac{x^4 - 16}{x + 2}$; 3. $\lim\limits_{x\to 1}\dfrac{x^2 - 1}{2x^2 - x - 1}$;

4. $\lim\limits_{x\to 2}(5x^3 + 7x)$; 5. $\lim\limits_{x\to 3}\dfrac{x^2 - 9}{x - 3}$; 6. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} - \lim\limits_{x\to 0}\dfrac{e^x - 1}{x}$

1.6. DERIVATIVES AND INTEGRALS

1.6.1. DERIVATIVES. GENERAL RULES

It's important to determine how quickly the function ***y*** is changing with variable ***x*** in many cases. The ***derivative*** of a function represents this information as an infinitesimal change in the function ***y* = *f*(*x*)** with respect to its variable ***x***.

Let's consider a function ***y* = *f*(*x*)** at two points with some values of argument x_0 and $x_0 + \Delta x$ (fig. 1.33). ***T***he difference between the points is an increment of an argument: $\Delta x = x - x_0$. The increment of function will be: $\Delta y = y - y_0$. For continuous functions, if $\Delta x \to 0$, then $\Delta y \to 0$. But it is impossible to foretell the value the attitude $\frac{\Delta y}{\Delta x}$ aspires at unlimited decrease $\mathbf{\Delta x}$, because it depends on a concrete kind of function $\boldsymbol{y(x)}$.

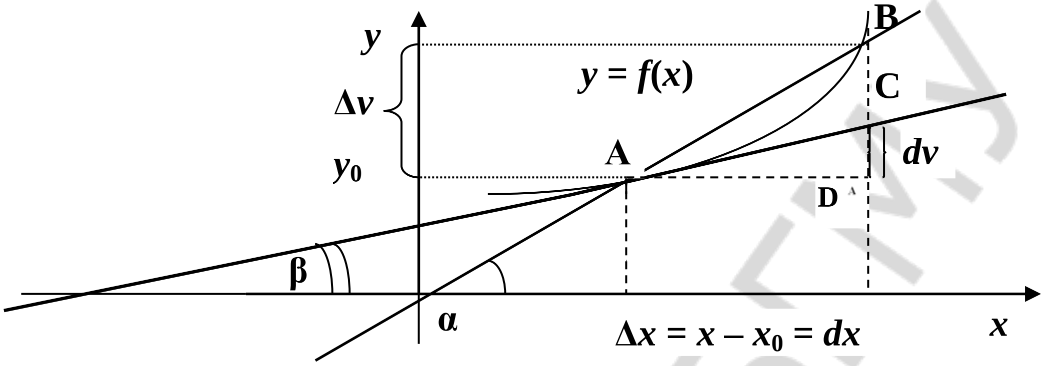


Fig. 1.33. Derivative of a function $y = f(x)$ — geometrical sence

Definition: Derivative of the function $\boldsymbol{y(x)}$ in the given point $\boldsymbol{x_0}$ is a limit of the attitude of an increment of function to an increment of argument at its unlimited decrease. Derivative of function of one argument is designated: $\boldsymbol{y'}$ or $\frac{dy}{dx}$. Thus: $y' = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, or $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Derivative of a function has simple *geometrical sense*.

From fig. 1.32 it is evident, that the attitude

$$\frac{\Delta y}{\Delta x} = \frac{BC}{AC} = \text{tg}\ \alpha,$$

where $\boldsymbol{\alpha}$ — a slope angle of the secant $\boldsymbol{AB}$ to an axis $\boldsymbol{OX}$.

If $\mathbf{\Delta x \to 0}$ the point $\boldsymbol{B}$ will move towards point $\boldsymbol{A}$, then $\mathbf{\Delta x}$ will unboundedly decrease and approach **0**, and the secant $\boldsymbol{AB}$ will approach the tangent $\boldsymbol{AC}$. Hence, a limit of the difference quotient is equal to a slope of a tangent at point $\boldsymbol{A}$.

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim \text{tg}\alpha = \text{tg}\beta.$$

Thus, derivative of a function at a point is numerically equal to the tangent of a corner between the tangent lead to the curve of function in the given point, and an axis $\boldsymbol{OX}$, — that is the ***geometrical sense*** of a derivative.

Derivative of a function has also a *mechanical sense or interpretation*.

Let's consider a movement of a material point along a coordinate line. The point displacement during the time interval from $\boldsymbol{t_0}$ till $\boldsymbol{t_0 + \Delta t}$ is equal to:

$$S(t_0 + \Delta t) - S(t_0) = \Delta S,$$

and its **average velocity** is: $v_{aver} = \frac{\Delta S}{\Delta t}$.

If $\Delta t \to 0$, an average velocity value approaches the certain value, which is called an **instantaneous velocity** $v(t_0)$ of a material point at the moment t_0.

$$v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt} = S'.$$

An **instantaneous velocity** $v_{\text{inst}} = v(t_0)$ is a derivative of a displacement with respect to time at the moment t_0:

$$v_{\text{inst}} = S'(t).$$

Similarly to this imagination, an **instantaneous** acceleration $a(t_0)$ is a derivative of a velocity with respect to time at the moment t_0:

$$a_{\text{inst}} = v'(t).$$

There are several certain rules how to calculate a derivative of elementary functions and its combinations.

Single derivatives of simple functions are calculated with the help of special ***table of derivatives*** (table 1.4). The derivatives of more complicated or composite functions are calculated easily using special ***differentiation rules***.

Table 1.4

Table of derivatives

Function	Derivative
x^a	ax^{a-1}
C (constant)	0
a^x	$a^x \ln a$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\text{tg}\, x$	$\frac{1}{\cos^2 x}$
$\text{ctg}\, x$	$-\frac{1}{\sin^2 x}$
$\text{arctg}\, x$	$\frac{1}{1+x^2}$

1.6.2. DIFFERENTIATION RULES

Constant Rule. The derivative of any constant C equals to zero: $C' = 0$.

Constant Multiple Rule. The derivative of constant C times a function is equal to the constant C times the derivative of the function:

$$(Cu)' = C \cdot (u)'.$$

Sum rule: the derivative of a sum is equal to the sum of the derivatives:

$$(u + v)' = u' + v'.$$

Product rule: the derivative of a product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first:

$$\boldsymbol{(u \cdot v)' = u' \cdot v + u \cdot v'.}$$

Quotient rule: the derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the square of the denominator:

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}.$$

Chain rule for a composite function:

Composite function consists of the combination of two (or more) functions.

Lets two functions $\boldsymbol{f(x)}$ and $\boldsymbol{g(x)}$ of the composite function $\boldsymbol{f(g(x))}$ are obtained by replacing each occurrence of $\boldsymbol{x}$ in $\boldsymbol{f(x)}$ by $\boldsymbol{g(x)}$. Thus, $\boldsymbol{f(g(x)) = f(u)}$, where $\boldsymbol{u = g(x)}$.

It is necessary to calculate a derivative of each function which is an argument of another function being a part of a composite function **in order to calculate a *composite function* derivative**. Then they should multiply each other:

$$\boldsymbol{f'(g(x)) = f'(u) \cdot u'(x).}$$

***Example 1*:** $y = \sin x$, and $x = (t^2 + 3t)$. Then $y = \sin(t^2 + 3t)$ is a composite function of x.

It's derivative is $y' = (\sin x)' \cdot (t^2 + 3t)' = \cos x \cdot (2t + 3) = (2t + 3) \cdot \cos(t^2 + 3t)$.

***Example 2*:** $y = \sin^3(\mathrm{tg}(x^2))$. It is necessary to make a number of the conditional steps-mental operations to find the derivative of this composite function.

1. To determine the *number* of functions-arguments which are the parts of a composite function.

1. The power function (cubic function) — the derivative is $3\sin^2(\mathrm{tg}(x^2))$.
2. $\sin(\mathrm{tg}(x^2))$ — derivative is equal to $\cos(\mathrm{tg}(x^2))$.
3. $\mathrm{tg}(x^2)$ — derivative is $\dfrac{1}{\cos^2 x^2}$.
4. x^2 — derivative is equal $2x$.

2. To combine the derivatives of the parts of a composite function:

$$y'_x = 3\sin^2(\mathrm{tg}(x^2)) \cdot \cos(\mathrm{tg}(x^2)) \cdot \frac{1}{\cos^2(x^2)} 2x.$$

Example 3. Calculate the function derivative y', if $y = \sin(\mathrm{tg}(\sqrt{x}))$, y is a composite function, $y' = (\sin(\mathrm{tg}(\sqrt{x})))' = \cos(\mathrm{tg}(\sqrt{x})) \cdot (\mathrm{tg}(\sqrt{x}))'$,· $(\mathrm{tg}(\sqrt{x}))'$ is also a composite function,

$$y' = \cos(\mathrm{tg}\sqrt{x}) \cdot \frac{1}{\cos^2\sqrt{x}} \cdot (\sqrt{x})'$$

$$y' = \cos(\mathrm{tg}\sqrt{x}) \cdot \frac{1}{\cos^2\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$y' = \cdot\frac{\cos(\mathrm{tg}\sqrt{x})}{2\sqrt{x}\,\cos^2\sqrt{x}}$$

EXERCISES

a. Calculate the derivatives of functions:

1. $y = 5x^3$;
2. $y = \sin^2 6x$;
3. $y = 3x^4 \cdot \ln 4x$;
4. $y = \sqrt[3]{x^4 + \sin^4 x}$;
5. $y = (\sin^2 x + 8x)^9$;
6. $y = (e^x + \sqrt{x}) \cdot \ln x$;
7. $y = \dfrac{\sin x}{\sqrt{x}}$;
8. $y = \ln(x^2 - 4x + 4)$.

1.6.3. MAXIMA AND MINIMA OF FUNCTIONS

The function $\boldsymbol{f(x)}$ has a **relative maximum** value at point $\boldsymbol{A}$, if $\boldsymbol{f(A)}$ is greater than any value in its immediate neighborhood.

The function $\boldsymbol{f(x)}$ has a **relative minimum** value at point $\boldsymbol{B}$, if $\boldsymbol{f(b)}$ is less than any value in its immediate neighborhood. At each of these points the tangent to the curve is parallel to the $\boldsymbol{x}$-axis so the derivative of the function is zero: $\boldsymbol{f'(x) = 0}$. The term local is used since these points are the maximum and minimum in this particular region. There may be others outside this region.

If $\boldsymbol{f(x)}$ has a local maximum or minimum at points $\boldsymbol{a}$ and $\boldsymbol{b}$, and if $\boldsymbol{f'(x)}$ exists, then $\boldsymbol{f'(x)} = 0$.

At points immediately to the left of a maximum the slope of the tangent is positive: $\boldsymbol{f'(x) > 0}$. While at points immediately to the right the slope is negative: $\boldsymbol{f'(x) < 0}$. In other words, at a maximum, $\boldsymbol{f'(x)}$ changes sign from + to –. At a minimum, $\boldsymbol{f'(x)}$ changes sign from – to + (fig. 1.34).

A point $\boldsymbol{x}$ at which the function has either a maximum or a minimum is called a **critical point**.

To find the maximum and minimum values of a function we need:

1. Solve the algebraic equation: $\boldsymbol{f'(x) = 0}$.

The roots $\boldsymbol{x_1, x_2, x_3 \ldots}$ of this equation are the stationary points.

2. Calculate the second derivative $\boldsymbol{f''(x)}$ and definite its sign at the points.

If the second derivative is positive at a stationary point is positive ($\boldsymbol{f''(x_1) > 0}$), the point $\boldsymbol{x_1}$ is a local **minimum**; if it is negative ($\boldsymbol{f''(x_2) < 0}$), the point $\boldsymbol{x_2}$ is a local **maximum**; if it is equal to zero: $\boldsymbol{f''(x_3) = 0}$, it may be no local extremums. In this case it is necessary to find a sign of the first derivative on the left side ($\boldsymbol{x < x_3}$) and on the right one ($\boldsymbol{x > x_3}$) from the $\boldsymbol{x_3}$. If the sing on the left side is "–" and on the right one is "+" there is a local **minimum** at the point $\boldsymbol{x_3}$. If the sing on the left side is "+" and on the right one is "–" there is a local **maximum** at the point $\boldsymbol{x_3}$. And if the sing not changes there is no extrema at this point.

3. Determine a value of function in points of maximum and minimum.

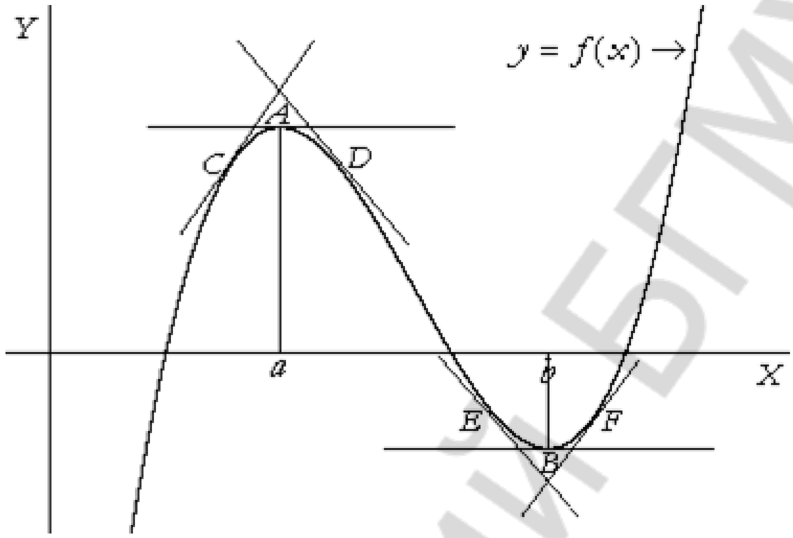


Fig. 1.34. The relative maximum and the relative minimum of the function $f(x)$

Example. Let $f(x) = 2x^3 – 9x^2 + 12x – 3$.

Are there any critical values — solutions to $f'(x) = 0$ — and do they determine a maximum or a minimum? And what are the coordinates on the graph of that maximum or minimum? Where are the turning points?

Solution. $f'(x) = 6x^2 – 18x + 12 = 6(x^2 – 3x + 2) = 6(x – 1)(x – 2) = 0$.

Implies: $x = 1$, $x = 2$.

Those are the critical values. Does each one determine a maximum or does it determine a minimum? To answer, we must evaluate the second derivative at each value.

$f'(x) = 6x^2 - 18x + 12$; $f''(x) = 12x – 18$ $f''(1) = 12 – 18 = –6$.

The second derivative is negative. The function therefore has a maximum at $x = 1$.

To find the y-coordinate — the extreme value — at that maximum we evaluate $f(1)$: $f(x) = 2x^3 – 9x^2 + 12x – 3$; $f(1) = 2 – 9 + 12 – 3 = 2$.

The maximum occurs at the point (1, 2).

Next, does $x = 2$ determine a maximum or a minimum?

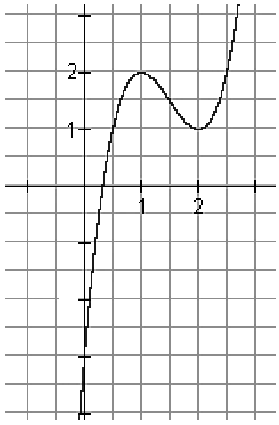
$f''(x) = 12x – 18$; $f''(2) = 24 – 18 = 6$.

The second derivative is positive. The function therefore has a minimum at $x = 2$.

To find the $\boldsymbol{y}$-coordinate — the extreme value — at that minimum, we evaluate $f(2)$: $f(x) = 2x^3 – 9x^2 + 12x – 3$ $f(2) = 16 – 36 + 24 – 3 = 1$.

The minimum occurs at the point (2, 1).

Here in fact is the graph of $f(x)$:



EXERCISES

a. Determine maxima and minima of functions:

1. $y = 2x^2 - x^4$; 2. $y = 2 + x - x^2$; 3. $y = \dfrac{x^3}{3} - x$; 4. $y = x \cdot e^{-x}$.

1.6.4. DIFFERENTIAL OF A FUNCTION

The **differential of a function** represents the principal part of the change in the function $\boldsymbol{y = f(x)}$ with respect to changes in the independent variable.

Definition. The differential $\mathbf{d}\boldsymbol{y}$ is defined as a product of function derivative $\boldsymbol{y'}$ and an increment (or differential) of an argument $\mathbf{d}\boldsymbol{x}$:

$$\mathbf{d}\boldsymbol{y} = \boldsymbol{y'} \cdot \mathbf{d}\boldsymbol{x}.$$

The argument differential $\mathbf{d}\boldsymbol{x}$ is equal to the increment of argument $\boldsymbol{\Delta x}$, i. e. $\mathbf{d}\boldsymbol{x} = \boldsymbol{\Delta x}$.

Differential $\mathbf{d}\boldsymbol{y}$ of function is not equal to its increment $\boldsymbol{\Delta y}$ but represents its main part: $\boldsymbol{\Delta y} \approx \mathbf{d}\boldsymbol{y} = \boldsymbol{y'} \cdot \mathbf{d}\boldsymbol{x}$.

In the fig. 1.31 differential $\mathbf{d}\boldsymbol{y}$ corresponds to line segment **CD**: $\mathbf{d}\boldsymbol{y} = \mathbf{[CD]}$.

1.6.5. INDEFINITE INTEGRALS. GENERAL RULES

We are able to find the function derivative $\boldsymbol{F'(x)}$ in any case. What about the reverse operation? Very often it is necessary to find some function $\boldsymbol{F(x)}$ the derivative of which is equal to the initial function $\boldsymbol{f(x) = F'(x)}$.

Definition. The function $\boldsymbol{F(x)}$ is called an antiderivative function of the initial function $\boldsymbol{f(x)}$, if the following equation is performed: $\boldsymbol{F'(x) = f(x)}$.

But, for example, if the derivative of expression $\boldsymbol{x^3 + 5}$ is $\boldsymbol{3x^2}$, an antiderivative of $\boldsymbol{3x^2}$ is $\boldsymbol{x^3 + 5}$. At the same time the derivative of $\boldsymbol{x^3 + 7}$ is also $\boldsymbol{3x^2}$, another antiderivative of $\boldsymbol{3x^2}$ is $\boldsymbol{x^3 + 7}$.

Similarly, another antiderivative of $\boldsymbol{3x^2}$ is $\boldsymbol{x^3 + 8}$ etc. In fact, every antiderivative of $\boldsymbol{3x^2}$ has the form $\boldsymbol{x^3 + C}$, where $\boldsymbol{C}$ is an **arbitrary constant**:

$$\boldsymbol{F'(x) = x^3 + C}.$$

Definition. Indefinite integral of a function ***f*(*x*)** is a set of all its antiderivatives ***F*(*x*)** of the initial function ***f*(*x*)**. The process of calculating an indefinite integral is called **integration**.

The symbol $\int f(x)\mathrm{d}x$ is used to indicate the indefinite integral of ***f*(*x*)**. The indefinite integral of any given function is not unique and can differ by up to a constant. Thus we write, $\int f(x)\mathrm{d}x = F(x) + C$,
where ***C*** is an arbitrary constant known as the **constant of integration**.

Features of the indefinite integral.

The integral of a sum or difference of functions is the sum or difference of the individual integrals. $\int (f \pm g)\mathrm{d}x = \int f\mathrm{d}x \pm \int g\mathrm{d}x$.

We can take the multiplicative constant **k** outside the integral sign.

$$\int \mathbf{k} y \mathrm{d}x = \mathbf{k} \int y \mathrm{d}x.$$

The main standard integrals are presented in table 6.1.

Table 6.1

Standard integrals

$\int 0 \cdot \mathrm{d}x = \mathrm{C}$	$\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \mathrm{tg}\,x + \mathrm{C}$
$\int 1 \cdot \mathrm{d}x = x + \mathrm{C}$	
$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + \mathrm{C} \quad (\alpha \neq -1)$	$\int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\mathrm{ctg}\,x + \mathrm{C}$
$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + \mathrm{C}$	$\int \frac{1}{\sin x}\,\mathrm{d}x = \ln\left\lvert \mathrm{tg}\frac{x}{2}\right\rvert + \mathrm{C}$
$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \begin{matrix} \arcsin x + C \\ -\arccos x + C \end{matrix}$	$\int \frac{1}{\cos x}\,\mathrm{d}x = \ln\left\lvert \mathrm{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\rvert + \mathrm{C}$
$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \begin{matrix} \mathrm{arctg}\,x + C \\ -\mathrm{arcctg}\,x + C \end{matrix}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \begin{cases} \arcsin \frac{x}{a} + C \\ -\arccos \frac{x}{a} + C \end{cases}$
$\int a^x dx = \frac{a^x}{\ln x} + \mathrm{C}$	$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \begin{cases} \frac{1}{a}\mathrm{arctg}\,\frac{x}{a} + C \\ -\frac{1}{a}\mathrm{arcctg}\,\frac{x}{a} + C \end{cases}$
$\int \sin x\,\mathrm{d}x = -\cos x + \mathrm{C}$	
$\int \cos x\,\mathrm{d}x = \sin x + \mathrm{C}$	
$\int \mathrm{tg}\,x\,\mathrm{d}x = -\ln\lvert\cos x\rvert + \mathrm{C}$	$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + \mathrm{C}$
$\int \mathrm{ctg}\,x\,\mathrm{d}x = \ln\lvert\sin x\rvert + \mathrm{C}$	

Sometimes it is impossible to find an antiderivative which is an elementary function. There are different methods of integration in this case. The simplest methods are linear integration and integration by substitution.

Linear integration allows us to break complicated integrals into simpler ones which correlate table integrals.

***Example*:** $\int(5x+\sin x)\mathrm{d}x=\int 5x\mathrm{d}x+\int \sin x\mathrm{d}x=\dfrac{5x^2}{2}-\cos x+C.$

Integration by substitution.

The **substitution rule** is an important tool for finding antiderivatives and integrals for composite function (like the chain rule for differentiation). It allows to involve new variables and its differentials with the main aim — to reduce previous variables for table standart integral with new variables.

***Example 1*.** Find the solution: $\int x\sqrt{x-1}\mathrm{d}x$

By using the substitution $t=\sqrt{x-1}$ we obtain

$$\int x\sqrt{x-1}\mathrm{d}x\left\|\begin{matrix} t=\sqrt{x-1}\Rightarrow t^2=x-1 \\ x=t^2+1\Rightarrow \mathrm{d}x=2t\mathrm{d}t\end{matrix}\right\|=\int(t^2+1)\cdot t\cdot 2t\mathrm{d}t=$$

$$=\int 2t^2(t^2+1)\mathrm{d}t=\int(2t^4+2t^2)\mathrm{d}t=$$

Then use linear integration $=\int 2t^4\mathrm{d}t+\int 2t^2\mathrm{d}t=\dfrac{2t^5}{5}+\dfrac{2t^3}{3}+\mathrm{C}$

Produce reverse substitution and final result is:

$$\int x\sqrt{x-1}\mathrm{d}x=\frac{2t^5}{5}+\frac{2t^3}{3}+\mathrm{C}=\frac{2}{5}\sqrt{(x-1)^5}+\frac{2}{3}\sqrt{(x-1)^3}+\mathrm{C}$$

***Example 2*.** Find the solution: $\int(1+\sin x)^3$.

By using the substitution $1+\sin x=t$ we obtain

$$\int(1+\sin x)^3\cos x\ \mathrm{d}x\left\|\begin{matrix} 1+\sin x=t \\ \cos x\ \mathrm{d}x=\mathrm{d}t\end{matrix}\right\|=\int t^3\mathrm{d}x=\frac{t^{3+1}}{3+1}+\mathrm{C}=\frac{t^4}{4}+\mathrm{C}$$

Produce reverse substitution and final result is:

$$\int(1+\sin x)^3\cos x\ \mathrm{d}x=\frac{(1+\sin x)^4}{4}+\mathrm{C}$$

EXERCISES

a. Calculate the integrals:

1. $\int 3x^3\mathrm{d}x$; 2. $\int(\sqrt{x}+\sin x+e^x)\mathrm{d}x$; 3. $\int\dfrac{\sin 2x}{\sin x}\mathrm{d}x$;

4. $\int e^{2x+1}\mathrm{d}x$; 5. $\int e^{\sin^2 x}\cdot\sin x\cdot\cos x\mathrm{d}x$; 6. $\int\dfrac{\mathrm{d}x}{x\cdot\ln x}$.

1.6.6. DEFINITE INTEGRAL

Integration was introduced as the reverse of differentiation. A more rigorous treatment would show that integration is a process of adding or «summation».

Consider the graph of the positive function $\boldsymbol{y(x)}$ shown in figure 1.35. Suppose we are interested in finding the area of the region bounded above by the graph of $\boldsymbol{y(x)}$, bounded below by the $\boldsymbol{x}$-axis, bounded to the left by the vertical line $\boldsymbol{x_1 = a}$, and bounded on the right by the vertical line $\boldsymbol{x_n = b}$.

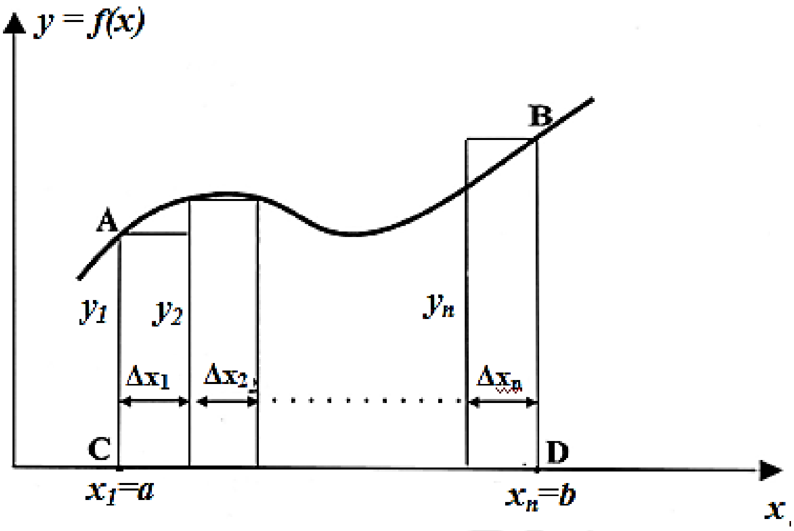


Fig. 1.35. Calculation of the area under a curve $f(x)$

One way in which this area can be approximated is to divide it into a number of rectangles, find the area of each rectangle, and then add up all these individual rectangular areas. The sum of the areas of all $\boldsymbol{n}$ rectangles is then

$$S_{ABDC} \approx \sum_{i=1}^{n} y_i \cdot \Delta x_i .$$

This quantity gives us an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. So, what we want to find is the value of this sum when $\boldsymbol{n}$ tends to infinity and $\boldsymbol{\Delta x}$ tends to zero. We write this value as

$$S_{ABCD} = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} y_i \Delta x_i = \int_a^b y \mathrm{d}x .$$

Limit of the sum is called ***the definite integral*** of $\boldsymbol{y}$ from $\boldsymbol{x = a}$ to $\boldsymbol{x = b}$ and it is written $\int_a^b y dx$.

Fundamental theorem of calculus (***the Newton–Leibniz formula***): Let $\boldsymbol{f(x)}$ be integrable over the interval **[a; b]**, and suppose there is an antiderivative $\boldsymbol{F(x)}$ of $\boldsymbol{f(x)}$ over the interval **[a; b]**. Then, the definite integral with integrand $\boldsymbol{f(x)}$ and limits $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the value of the antiderivative $\boldsymbol{F(b)}$ minus the value of antiderivative $\boldsymbol{F(a)}$:

$$\int_a^b f(x)\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

The notation $F(x)\Big|_a^b$ means the following: at first substitute the upper limit ***b*** into the function ***F*(*x*)** to obtain ***F*(*b*)** and then from ***F*(*b*)** we subtract ***F*(*a*)**, the value obtained by substituting the lower limit ***a*** into ***F*(*x*)**. This ***Newton–Leibniz formula*** (1.11) allows us to easily solve definite integral, if we can find the antiderivative function of the integrand.

Unlike the ***indefinite integral***, which is ***the set of functions***, the ***definite integral*** is ***a numerical value***, that represents the area under the curve ***f*(*x*)**. ***Features of the definite integral*:**

1. $\int\limits_a^b f(x)\mathrm{d}x = -\int\limits_b^a f(x)\mathrm{d}x.$
2. $\int\limits_a^b (f(x) + g(x))\mathrm{d}x = \int\limits_a^b f(x)\mathrm{d}x + \int\limits_a^b g(x)\mathrm{d}x.$
3. $\int\limits_a^a f(x)\mathrm{d}x = 0.$
4. $\int\limits_a^b kf(x)\mathrm{d}x = k\int\limits_a^b f(x)\mathrm{d}x.$
5. $\int\limits_a^b f(x)\mathrm{d}x = \int\limits_a^c f(x)\mathrm{d}x + \int\limits_c^b f(x)\mathrm{d}x.$

Calculation of the area between two curves $y_1(x)$ and $y_2(x)$.

The definite integral is used for calculation of the area between two curves $y_1(x)$ and $y_2(x)$.

At first we must find the crossing points of this curves by solving the equation: $y_1(x) = y_2(x)$. If this points are x_1 and $x_2 > x_1$, we can calculate the area between the curves (we consider $y_1(x) > y_2(x)$ at this region):

$$S = \int\limits_{x_1}^{x_2} (y_1 - y_2)\mathrm{d}x.$$

Example 1.

Consider the integral $\int\limits_0^3 x\mathrm{d}x$. The area under the line is the triangle (fig. 1.36). The area of any triangle is half its base times the height. It is: $S = \frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}$.

As expected, the integral yields the same result:

$$\int\limits_0^3 x\mathrm{d}x = \frac{x^2}{2}\Bigg]_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2} - 0 = \frac{9}{2}.$$

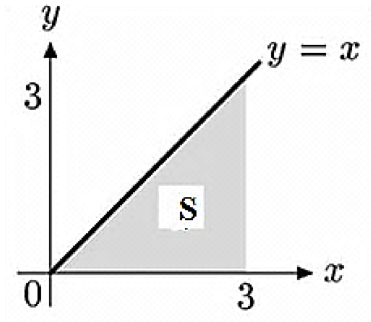


Fig. 1.36. The area *S* under the line ***y* = *x***

Example 2.

Calculate the area S limited the curve $y = x^2$, axis x and lines $x_1 = -1$ и $x_2 = 2$. In fig. 1.37 this area is cross-hatched.

$$S = \int_{-1}^{2} x^2 \mathrm{d}x = \frac{x^3}{3}\Bigg]_{-1}^{2} = \frac{8}{3} - \left(-\frac{1}{3}\right) = \frac{9}{3} = 3.$$

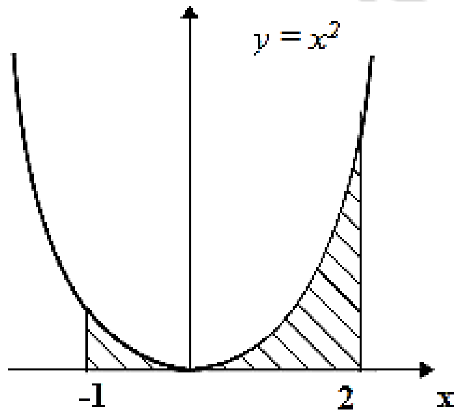


Fig. 1.37. The area **S** under the curve ***y* = *x*^{2}**

Example 3. Calculate the integral

$$\int_{-2}^{2} (4 - x^2)\mathrm{d}x = (4x - \frac{x^3}{3})\Bigg|_{-2}^{2} = (4 \cdot 2 - \frac{8}{3}) - (-4 \cdot 2 + \frac{8}{3}) = 2(8 - \frac{8}{3}) = \frac{32}{3} = 10\frac{2}{3}.$$

Exercises

a. Calculate the definite integrals:

1) $\int_{4}^{9} \sqrt{x}\,\mathrm{d}x$; 2) $\int_{0}^{\pi} \sin x\,\mathrm{d}x$; 3) $\int_{0}^{4} x\sqrt{x^2 + 9}\,\mathrm{d}x$;

4) $\int_{0}^{1} \frac{x^3 \mathrm{d}x}{3 + x^4}$; 5) $\int_{0}^{4} \frac{x\,\mathrm{d}x}{\sqrt{x^2 + 9}}$; 6) $\int_{-1}^{0} x^5 (1 - x^6)^7 \mathrm{d}x$.

b. Calculate the area between two curves:

1) $y = 2x - x^2$, $y = x$; 2) $y = x^2$, $y = -2x$.

THE BASICS OF PHYSICS

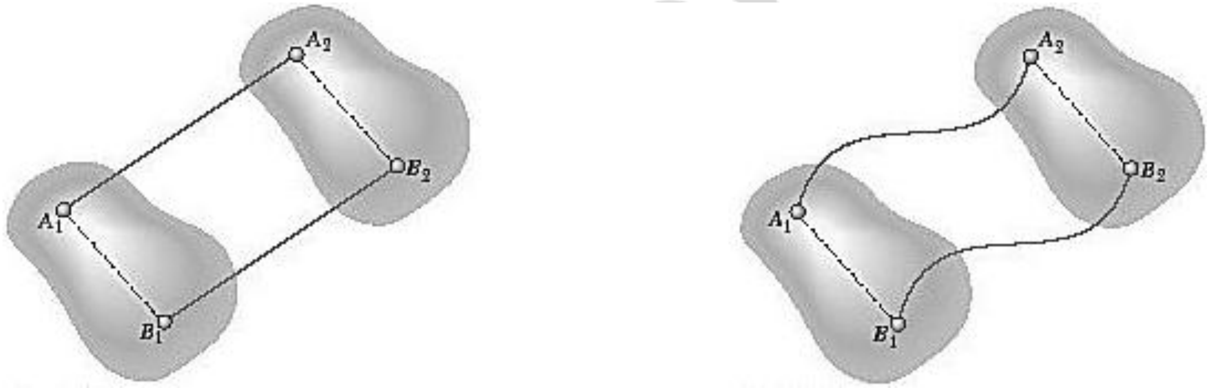
2. KINEMATICS

The study of the object motion, and the related concepts of force and energy, forms the field of science called mechanics. Mechanics is divided into three parts: kinematics, dynamics, and statics. Kinematics describes how objects move, dynamics deals with force and why objects move. Statics is concerned with the analysis of loads (i. e. forces) acting on and within physical systems that are in equilibrium.

Kinematics is the branch of mechanics which deals with the study of the motion without taking into account the factors responsible for producing motion.

In mechanics the concept (or model) of an idealized particle is used. It is considered to be a mathematical point (point particle) with no spatial extent (no size). The particle model is useful in many real situations where we are interested only in translational motion.

Translational motion means motion in which all particles in the body move along parallel paths, and with the same velocity and acceleration (fig. 2.1).



a ***b***

Fig. 2.1. Rectilinear (*a*) and curvilinear (*b*) rigid body translation

Therefore the object's size is not significant in case of translational motion. For example, we might consider a billiard ball, or even a spacecraft traveling toward the Moon, as a point particle, for many purposes.

There are two types of translational motion: rectilinear translation (paths are straight lines) and curvilinear translation (paths are curved, e. g. circular) (fig. 2.1).

In this section we consider the simplest two cases of translational motion: a body motion along a straight line (linear or one-dimensional motion) and curvilinear motion in a circle path.

2.1. MECHANICAL MOTION CHARACTERISTICS

The main task of kinematics is to determine a position of the body at any instant of time. Any measurement of a body position must be made with respect to a reference frame. In mechanics a frame of reference is represented by a set of coordinate axes (fig. 2.2), therewith the origin (or zero point) of the frame is often chosen as a *reference point*. To locate a body means to find its coordinates (x, y) relative to the origin of coordinate axes (to the reference point). Any point on the plane can be specified by giving its x and y coordinates (point A in fig. 2.2). In three dimensions, a z axis perpendicular to the x and y axes is added.

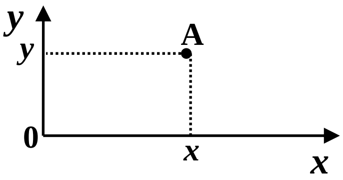


Fig. 2.2. Position of a point A on the plane (x, y)

For one-dimensional motion we usually choose the x axis as the line along which the motion takes place (fig. 2.3). Then the position of a point at any moment is given by its x coordinate. If the motion is vertical, as with a falling object, we usually use the y axis.

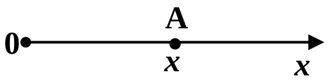


Fig. 2.3. Position of a point A for one-dimensional motion

The position of the moving object is changed with time and hence the x coordinate is a function of time t ($x = f(t)$). The dependence $x = f(t)$ is called *the equation of motion*.

To describe the motion, such physical characteristics as *path, distance, displacement, velocity and acceleration are introduced*.

Path (trajectory) is the curve along which the object moves (line ACDB in fig. 2.4).

The distance **(s)** *is the actual length of the path followed by the moving object (length of the line ACDB). It is a scalar quantity.*

We need to make a distinction between *the distance* a body has travelled and its *displacement*. *Displacement is defined as the change in position of the object*, i. e. it shows how far the object is from its starting point (vector $\boldsymbol{r}$ in fig. 2.4). *The direction of the displacement vector is from an original position to a final position*. Displacement does not depend on the actual path followed by the object. Only the initial and the final positions determine the displacement.

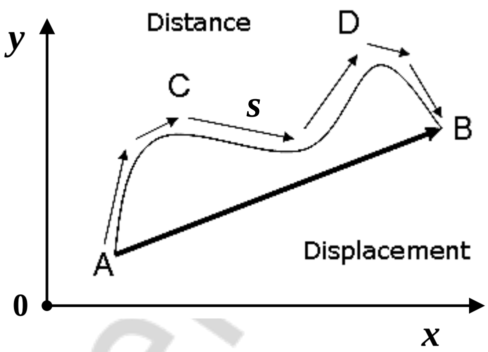


Fig. 2.4. Distance s (length of the line *ACDB*) and the displacement vector

Let us consider linear motion of an object during a particular time interval. Suppose that at some initial time t_1, the object is on the x axis at the position x_1

in the coordinate system shown in fig. 2.5. At some later time, $\boldsymbol{t_2}$, suppose the object has moved to position $\boldsymbol{x_2}$. The displacement vector $\boldsymbol{r}$ shows a change in the position of the object along the $\boldsymbol{x}$ axis during time interval $\boldsymbol{\Delta t = t_2 - t_1}$.

From fig. 2.5 it is seen that the magnitude of the displacement vector $\boldsymbol{r}$ is equal to the change in the $\boldsymbol{x}$ coordinate of the object: $\boldsymbol{r = \Delta x = x_2 - x_1}$. The distance travelled $\boldsymbol{s}$ is equal to the magnitude of displacement: $\boldsymbol{r = \Delta x = x_2 - x_1 = s}$.

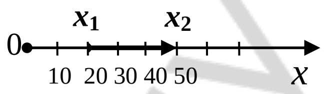


Fig. 2.5. The displacement vector

Note that the distance and the magnitude of the displacement vector are equal only in case of linear motion in the same direction. In all other cases $\boldsymbol{r < s}$ (fig. 2.4).

The SI units for the displacement and the distance are meters (*m*).

Velocity vector $\boldsymbol{\upsilon}$ *describes how fast and in what direction the body moves.*

Acceleration vector $\boldsymbol{a}$ *describes how fast and in what direction the velocity of the body changes.*

***Example 2.1*. Distance and displacement.**

A car moves along a circular path from point *A* to point *B* (fig. 2.6). If *AB* is the diameter of the circle, find (a) the magnitude of the displacement and (b) the total distance travelled.

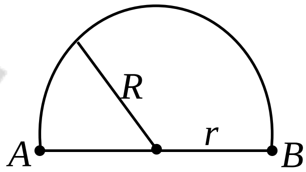


Fig. 2.6. Example 2.1

Solution. The magnitude of the displacement r is equal to the diameter of the circle: $r = AB = 2R$.

Distance s is equal to the half of the circle length: $s = \pi R$.

***Example 2.2*. Distance and displacement.**

The object starts from the ground (point *A*) and moves vertically upwards to a maximum height of h (point *B*) and falls back to the ground (fig. 2.7). Find (a) the magnitude of the displacement and (b) the total distance travelled.

Solution. The displacement is equal to zero and the total distance travelled is equal to $2h$ ($s = 2h$).

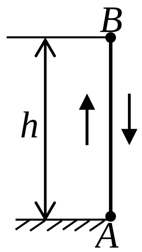


Fig. 2.7. Example 2.2

2.2. UNIFORM LINEAR MOTION

Uniform linear motion is a body motion along a straight line with constant velocity.

Velocity. *Velocity* $\boldsymbol{\upsilon}$ *is defined as the rate of a body position change in a particular direction with respect to time:*

$$\vec{\boldsymbol{\upsilon}} = \frac{\vec{\boldsymbol{r}}}{\boldsymbol{t}}. \tag{2.1}$$

Velocity vector has the same direction as the displacement vector $\boldsymbol{r}$. It is a vector quantity. If the body travels in the same direction and covers equal distances during equal time intervals, then its velocity is said to be *a uniform velocity*.

In physics when solving many problems, the magnitudes of various physical vector quantities should be frequently determined. These magnitudes are always determined by finding the projection of the vectors on the chosen coordinate axes.

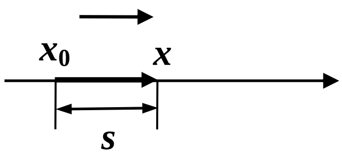


Fig. 2.8. The velocity vector

Let a particle be in a linear motion with a uniform velocity (fig. 2.8). We choose positive direction of the x axis along the motion direction. A particle is at the position $\boldsymbol{x_0}$ at initial instant of time $\boldsymbol{t_0 = 0}$ and at the position $\boldsymbol{x}$ at instant of time $\boldsymbol{t}$. From fig. 2.8 it is seen that $\boldsymbol{x}$ component of the velocity and the displacement vectors are equal to their magnitudes: $\boldsymbol{\upsilon_x = \upsilon}$, $\boldsymbol{r_x = r}$. Then the magnitude of the velocity vector is given by

$$\boldsymbol{\upsilon = \frac{r}{t}}. \tag{2.2}$$

Taking into account that the magnitude of the displacement vector $\boldsymbol{r}$ is equal to the distance ($\boldsymbol{s = x - x_0}$) we obtain that the total distance travelled is equal to magnitude of the velocity vector multiplied by the total elapsed time:

$$\boldsymbol{r = s = x - x_0 = \upsilon \cdot t}. \tag{2.3}$$

From the Eq. 2.3 it follows that the position of a particle $\boldsymbol{x}$ at any instant of time is defined as:

$$\boldsymbol{x = x_0 + \upsilon \cdot t}. \tag{2.4}$$

This is ***the basic equation of a uniform motion.***

Figure 2.9 shows a graph of the magnitude of velocity $\boldsymbol{\upsilon}$ and the distance $\boldsymbol{s}$ of a particle versus the time in case of a uniform motion. The slope of the $\boldsymbol{s(t)}$ graph is defined by the velocity magnitude, i. e. tan $\boldsymbol{\alpha = \upsilon}$. The larger the velocity, the larger the angle $\boldsymbol{\alpha}$ formed by the $\boldsymbol{s(t)}$ graph with the $\boldsymbol{t}$ axis.

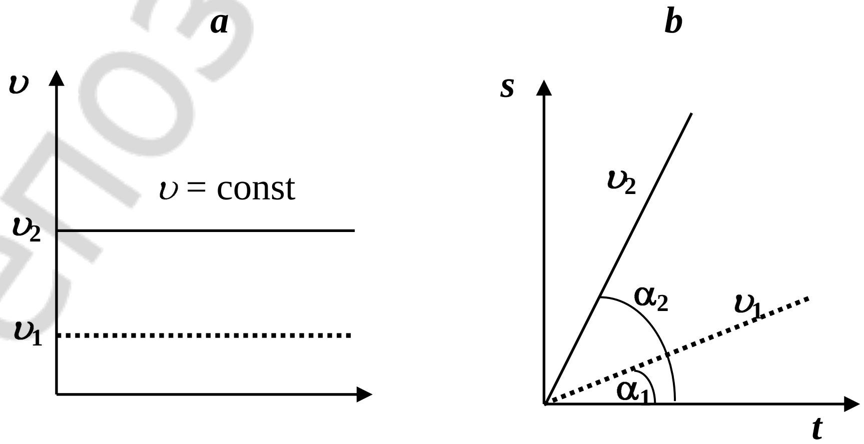


Fig. 2.9. A plot of the magnitude of velocity υ (*a*) and the distance $\boldsymbol{s}$ (*b*) of a particle versus time for case of a uniform linear motion

The SI units of velocity are meter per second (m/s).

***Example 2.3*. Distance.**

A car travels along a straight road with constant velocity 80 km/hr. How far does the car travel after 30 min?

***Solution*.** For solving a problem we use Eq. 2.3. The time interval t = 30 min = 0.5 hr. The distance travelled at υ = 80 km/hr is

$$s = \upsilon \cdot t = (80 \text{ km/hr})\,(0.5 \text{ hr}) = 40 \text{ km}.$$

***Example 2.4*. Equation of motion.**

A car's position as a function of time is given by $x = 5 + 30 \cdot t$. What are (a) the coordinate, (b) its displacement and distance travelled after 10 s?

***Solution*.**

a) Setting t = 10 s in the equation of motion gives the coordinate x

$$x = 5 + 30 \cdot t = 5 \text{ m} + 30 \text{ m/s} \cdot 10 \text{ s} = 305 \text{ m}.$$

b) In case of uniform linear motion in the same direction the magnitude of the displacement vector and the distance are equal and are defined by Eq. 2.3. Then displacement (distance) during time interval t = 10 s is

$$r = s = x - x_0 = 30 \cdot t = 30 \text{ m/s} \cdot 10 \text{ s} = 300 \text{ m}.$$

2.3. NON-UNIFORM LINEAR MOTION

A body has *a non-uniform motion* if it travels unequal distances during equal time intervals.

2.3.1. AVERAGE AND INSTANTANEOUS VELOCITY AND SPEED

If the body covers unequal distances during equal time intervals, then velocity is called a variable velocity. In case of a non-uniform motion the average velocity $\boldsymbol{\upsilon}_{\mathbf{av}}$ during some time interval is defined as

$$\overrightarrow{\boldsymbol{\upsilon}_{\mathbf{av}}} = \frac{\vec{\boldsymbol{r}}}{\Delta \boldsymbol{t}}, \qquad (2.5)$$

where $\boldsymbol{r}$ is the displacement, $\Delta \boldsymbol{t}$ is the time interval.

Magnitude of the average velocity vector is given by

$$\boldsymbol{\upsilon}_{\mathbf{av}} = \frac{\boldsymbol{r}}{\Delta \boldsymbol{t}}, \qquad (2.6)$$

where r is the magnitude of the displacement vector, $\Delta \boldsymbol{t}$ is the time interval.

The average speed of a body $\boldsymbol{\upsilon}_s$ is the total distance travelled divided by the time interval $\Delta \boldsymbol{t}$:

$$\boldsymbol{\upsilon}_s = \frac{\boldsymbol{s}}{\Delta \boldsymbol{t}}. \qquad (2.7)$$

Average speed is a scalar and it is not the magnitude of average velocity vector.

Instantaneous velocity (i. e. the velocity at a specific instant of time or specific point along the path) is the limit of the average velocity as the time interval approaches to zero; it is equal to the instantaneous rate of a body position change with time.

If $\boldsymbol{\Delta t \to 0}$, then from the Eq. 2.5 it follows that instantaneous velocity is defined as

$$\vec{\upsilon} = \lim_{\Delta t \to 0} \frac{\vec{r}}{\Delta t}. \qquad (2.8)$$

The magnitude of the instantaneous velocity is

$$\upsilon = \lim_{\Delta t \to 0} \frac{r}{\Delta t}. \qquad (2.9)$$

The magnitude of the instantaneous speed is always equal to the magnitude of the instantaneous velocity because the distance travelled and the magnitude of displacement vector become the same when they are infinitesimally small. Then we can write

$$\upsilon = \lim_{\Delta t \to 0} \frac{r}{\Delta t} = \lim_{\Delta t \to 0} \frac{s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \qquad (2.10)$$

where $\frac{dx}{dt}$ is the derivative of $\boldsymbol{x}$ with respect to time.

Note that in case of uniform linear motion in the same direction the magnitude of the velocity vector is speed.

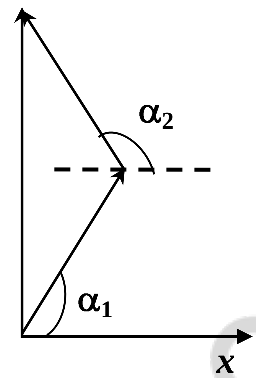


Fig. 2.10. Example 2.5

***Example 2.5*. The average speed and the average velocity.**

A particle moves half of its distance under the angle $\alpha_1 = 45°$ with respect to the *x*-axis at velocity $\upsilon_1 = 5$ m/s and the other half of its distance with the velocity $\upsilon_2 = 10$ m/s under the angle $\alpha_2 = 135^o$ (fig. 2.10). (a) What is the average speed of the particle? (b) What is the average velocity?

Solution.

a) For solving a problem we use Eqs. 2.3, 2.6 and 2.7. The time elapsed to cover the first half of the distance

$$t_1 = \frac{s}{2\upsilon_1}.$$

The time elapsed to cover the second half of the distance is $t_2 = \frac{s}{2\upsilon_2}$.

The time taken to cover total distance is

$$\Delta t = t_1 + t_2 = \frac{s}{2\upsilon_1} + \frac{s}{2\upsilon_2} = \frac{s}{2} \cdot \frac{\upsilon_1 + \upsilon_2}{\upsilon_1 \upsilon_2}.$$

Then the average speed is

$$\upsilon_s = \frac{s}{\Delta t} = \frac{s}{\frac{s}{2}\cdot\frac{\upsilon_1+\upsilon_2}{\upsilon_1\upsilon_2}} = \frac{2\upsilon_1\upsilon_2}{\upsilon_1+\upsilon_2} = \frac{2(5\text{ m/s})\cdot 10\text{ (m/s)}}{15\text{ m/s}} = 6.7\text{ m/s}.$$

The average speed is 6.7 m/s.

b) The magnitudes of displacement vectors r_1 and r_2 are $r_1 = \upsilon_1 t_1 = s/2$ and $r_2 = \upsilon_2 t_2 = s/2$, respectively. The displacement vectors r, r_1 and r_2 form rectangular triangle as shown in fig. 2.10, from which it follows

$$r = \sqrt{r_1^2 + r_1^2} = \frac{s}{\sqrt{2}}.$$

Then the magnitude of average velocity is

$$\upsilon_{\text{av}} = \frac{r}{\Delta t} = \frac{s}{\sqrt{2}\Delta t} = \frac{\upsilon_s}{\sqrt{2}} = \frac{6.7\text{ m/s}}{\sqrt{2}} = 4.8\text{ m/s}.$$

The average velocity is 4.8 m/s. The values of average speed and average velocity differ.

NOTE. *The difference between the average speed and the magnitude of the average velocity can occur when the motion is not in the same direction*.

***Example 2.6*. The average speed and the average velocity.**

An automobile travels on a straight road for 40 km at 80 km/h. It then continues in the same direction for another 20 km at 60 km/h. (a) What is the average velocity of the car during the full trip? (b) What is the average speed?

Solution. A car travels in the same direction, then the magnitude of its displacement is equal to the distance, i. e. $\boldsymbol{r} = \boldsymbol{s}$. For solving a problem we use Eqs. 2.3 and 2.7.

Solving Eq. 2.3 for t and setting υ_1 = 80 km/hr, s_1 = 40 km and υ_2 = 60 km/hr, s_2 = 20 km gives the time elapsed to cover the first (t_1) and the second (t_2) part of the distance, respectively

$$t_1 = \frac{s_1}{\upsilon_1} = \frac{40}{80} = 0.5\text{ hr},$$

$$t_2 = \frac{s_2}{\upsilon_2} = \frac{20}{60} = 0.3\text{ hr}.$$

The total time taken to cover total distance is

$$\Delta t = t_1 + t_2 = 0.5 + 0.3 = 0.8\text{ hr}.$$

The total distance travelled is

$$s = 40 + 20 = 60\text{ km}.$$

Setting s = 60 km and Δt = 0.8 hr in Eq. 2.7 we obtain

$$\upsilon_{\text{av}} = \frac{r}{\Delta t} = \frac{s}{\Delta t} = \frac{60\text{ km}}{0.8\text{ hr}} = 75\text{ km/hr}.$$

Both the average velocity and speed have the same value 75 km/hr.

NOTE. *Average speed and average velocity have the same magnitude when the motion is in the same direction*.

***Example 2.7*. Average velocity.**

A person walks 50 m at a velocity 1 m/s and then run 60 m for 20 s along a straight track. What is the average velocity of a person?

Solution. For solving a problem we use Eqs. 2.3 and 2.6 ($\boldsymbol{r} = \boldsymbol{s}$).

Solving Eq. 2.3 for t and setting υ_1 = 1.0 m/s, s_1 = 50 m gives the time elapsed to cover the first part of the distance

$$t_1 = \frac{s_1}{\upsilon_1} = \frac{50\text{ m}}{1\text{ m/s}} = 50\text{ s}.$$

The total time taken to cover total distance is

$$t = t_1 + t_2 = 50\ s + 20\ s = 70\text{ s}.$$

The total distance travelled is

$$s = 50\text{ m} + 60\text{ m} = 110\text{ m}.$$

Setting s = 110 km and t = 70 s in Eq. (2.6) gives:

$$\upsilon_{\text{av}} = \frac{r}{\Delta t} = \frac{s}{\Delta t} = \frac{110\text{ m}}{70\text{ s}} = 1.6\text{ m/s}.$$

The average velocity is 1.6 m/s.

2.3.2. AVERAGE AND INSTANTANEOUS ACCELERATION

The velocity generally changes with time either in magnitude or in direction or in both. In this case the motion of a body is said to be accelerated (or retarded). The change of velocity with time is measured by a vector quantity called acceleration $\boldsymbol{a}$.

If velocity changes by unequal amounts during equal time intervals, the body has a variable acceleration. In case of non-uniformly accelerated motion the term of the average acceleration is defined. If $\boldsymbol{\upsilon_1}$ is the velocity at any time $\boldsymbol{t_1}$ and $\boldsymbol{\upsilon_2}$ is the velocity at another time $\boldsymbol{t_2}$, then $\Delta\boldsymbol{\upsilon} = \boldsymbol{\upsilon_2} - \boldsymbol{\upsilon_1}$ is the change in velocity, $\boldsymbol{\Delta t = t_2 - t_1}$ is the time elapsed. Average acceleration is defined as the ratio of the change in velocity to the time interval:

$$\overrightarrow{\boldsymbol{a}_{\mathbf{av}}} = \frac{\overrightarrow{\Delta\boldsymbol{\upsilon}}}{\Delta\boldsymbol{t}}. \qquad (2.11)$$

Instantaneous acceleration is defined as the limit of the average acceleration as the time interval approaches to zero. If $\boldsymbol{\Delta t \to 0}$, then from the equation 2.11 it follows that instantaneous acceleration is equal to

$$\vec{\boldsymbol{a}} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta\boldsymbol{\upsilon}}}{\Delta\boldsymbol{t}}. \qquad (2.12)$$

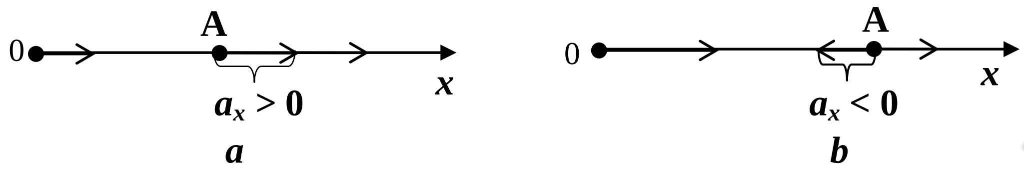


Fig. 2.11. Projections of the acceleration vector $\boldsymbol{a}$ along the $\boldsymbol{x}$ axis for case of an accelerated (a) and retarded (b) linear motion

Magnitude of the instantaneous acceleration vector is given by

$$\boldsymbol{a} = \lim_{\Delta t \to \infty} \frac{\Delta \upsilon}{\Delta t} = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right). \qquad (2.13)$$

where $\frac{d\upsilon}{dt}$ is the derivative of the magnitude of the velocity with respect to time, $\frac{d^2x}{dt^2}$ is the second derivative of $\boldsymbol{x}$ with time.

Units of acceleration are meter per second squared (m/s^2).

***Example 2.8*. Average acceleration.**

A car accelerates along a straight road from rest to 90 km/h during time interval equal to 5 s. What is the magnitude of its average acceleration?

Solution. We use Eq. 2.11, where we set Δt = 5 s. The car starts from rest, so υ_1 = 0 m/s. The final velocity is υ_2 = 90 km/h = 90 · 10^3 m/3600 s = 25 m/s.

From Eq. 2.11 the average acceleration is

$$a_{av} = \frac{\upsilon_2 - \upsilon_1}{\Delta t} = \frac{25 \text{ m/s} - 0 \text{ m/s}}{5 \text{ s}} = 5 \text{ m/s}^2.$$

2.4. UNIFORMLY ACCELERATED LINEAR MOTION

If the body travels in some direction and its velocity changes by equal amounts during equal time intervals, however small these intervals may be, then its acceleration is said to be *a uniform acceleration.*

In case of motion with a uniform acceleration, the acceleration $\boldsymbol{a}$ is

$$\vec{a} = \frac{\vec{\upsilon} - \vec{\upsilon}_0}{t}, \qquad (2.14)$$

where $\boldsymbol{\upsilon_0}$ and $\boldsymbol{\upsilon}$ are the initial and the final velocity of the object, respectively, $\boldsymbol{t}$ is time interval.

The velocity of the particle that has been accelerated after some elapsed time $\boldsymbol{t}$ *is*

$$\upsilon = \upsilon_0 + \boldsymbol{a} \cdot \boldsymbol{t}. \qquad (2.15)$$

In this section we consider the case of linear motion. Let us choose the positive direction of $\boldsymbol{x}$ axis coinciding with the direction of the initial velocity $\boldsymbol{\upsilon_0}$ (fig. 2.11). To determine the magnitudes of the acceleration and the velocity vectors, let us find the projections of these vectors along x axis. If the direction of the acceleration vector coincides with the direction of the initial velocity the projection of the vector $\boldsymbol{a}$ along x axis is equal to $+a$ ($a_x = +a$) (a is the magnitude of the vector $\boldsymbol{a}$), and $a_x = -a$ when the directions of the vectors $\boldsymbol{a}$ and υ_0 are opposite (fig. 2.11).

Thus the magnitude of the acceleration vector is defined as:

$$a_x = \frac{\upsilon_x - \upsilon_{ox}}{t}, \tag{2.16}$$

where $a_x = \pm a$ (a is the magnitude of the vector $\boldsymbol{a}$).

If the velocity increases with time, the acceleration is positive ($a > 0$), but if the velocity decreases, the acceleration is negative ($a < 0$) and it is called the retardation (motion is decelerated or retarded).

*The magnitude of the velocity of the uniformly accelerated particle after some elapsed time **t** is given by*

$$\upsilon_x = \upsilon_{0x} + a_x \cdot t. \tag{2.17}$$

For case of linear motion $\upsilon_x = \upsilon$, $\upsilon_{0x} = \upsilon_0$, that is,

$$\upsilon = \upsilon_0 \pm a \cdot t, \tag{2.18}$$

where υ_0 and υ are the initial and the final speed of the particle, respectively.

As a check, note that this equation reduces to $\upsilon = \upsilon_0$ at $t = 0$. As a further check, take the derivative of the $\upsilon(t)$ function. Doing so yields $\frac{d\upsilon}{dt} = a$, which is the definition of a.

Figure 2.12, *a* shows a plot of the $\upsilon(t)$ function defined by Eq. 2.18.

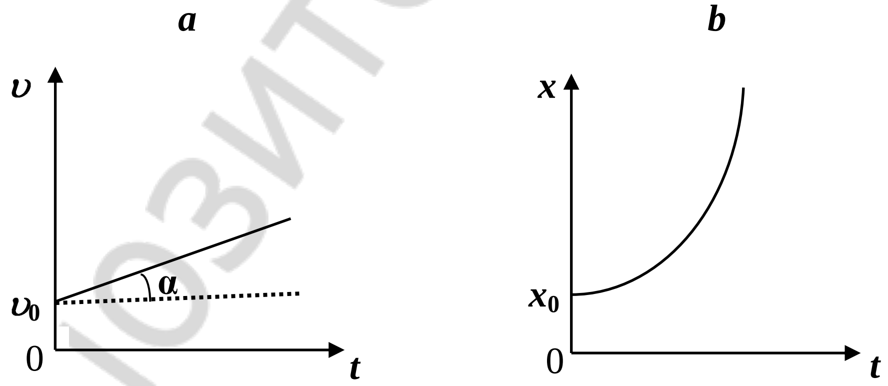


Fig. 2.12. A plot of the speed υ (*a*) and the coordinate x (*b*) of an object moving with constant acceleration versus time

The function is linear and thus the plot is a straight line. The slope of the graph is defined by the acceleration magnitude, i. e. tan $\alpha = a$.

*Position (coordinate **x**) of the particle moving with a uniform acceleration at any instant of time is given by the equation:*

$$x = x_0 + \upsilon_0 t \pm \frac{at^2}{2}. \tag{2.19}$$

This is ***the basic equation of uniformly accelerated motion***. The function defined by Eq. 2.19 is quadratic and thus the plot is curved (fig. 2.12, *b*).

As a check, note that putting $\boldsymbol{t = 0}$ yields $\boldsymbol{x = x_0}$. As a further check, taking the derivative of the function $\boldsymbol{x(t)}$ given by Eq. 2.19 yields to Eq. 2.18.

Distance travelled by a uniformly accelerated particle by a given time.

$$\boldsymbol{s = x - x_0 = \upsilon_0 t \pm \frac{at^2}{2}}. \tag{2.20}$$

By substituting $\boldsymbol{t = (\upsilon - \upsilon_0)/a}$ from the Eq. 2.18 into the equation 2.20, we obtain *speed of a particle after covering a certain distance.*

$$\boldsymbol{\upsilon^2 = \upsilon_0^2 \pm 2as}, \tag{2.21}$$

where $\boldsymbol{\upsilon_0}$ is the initial speed, $\boldsymbol{s}$ is the distance. This equation is useful if we do not know $\boldsymbol{t}$ and are not required to find it. It can be given in other form:

$$\boldsymbol{2as = \upsilon^2 - \upsilon_0^2}. \tag{2.21′}$$

However other equations can be derived that might be useful in certain situations. Thus two equations 2.18 and 2.19 can be combined to yield two additional equations of linear motion with constant acceleration, each of which involves a different “missing variable”.

1. We can eliminate the acceleration $\boldsymbol{a}$ to produce the following equation

$$\boldsymbol{x = x_0 + \frac{1}{2}(\upsilon_0 \pm \upsilon)t}. \tag{2.22}$$

2. Finally, we can eliminate $\boldsymbol{\upsilon_0}$, obtaining

$$\boldsymbol{x = x_0 + \upsilon t \pm \frac{at^2}{2}}. \tag{2.23}$$

Table 2.1 lists the basic equations of motion with constant acceleration and the specialized equations which have been derived above. To solve a problem you can choose an equation for which the only unknown variable is the variable requested in the problem.

Table 2.1

Equations of Motion with Constant Acceleration

Equation	Missing quantity	Equation Number
$\upsilon = \upsilon_0 \pm at$	$x - x_0$	2.18
$x = x_0 + \upsilon_0 t \pm \frac{at^2}{2}$	υ	2.19
$s = \upsilon_0 t \pm \frac{at^2}{2}$	υ	2.20
$\upsilon^2 = \upsilon_0^2 \pm 2as$	t	2.21
$x = x_0 + \frac{1}{2}\ \upsilon_0 \pm \upsilon\ t$	a	2.22
$x = x_0 + \upsilon t \pm \frac{at^2}{2}$	υ_0	2.23

***Example 2.9*. Velocity and acceleration.**

A particle's position on the x axis is given by

$$x = 5 + 10t + 3t^2,$$

where x in meters and t in seconds. Find (a) the particle's instantaneous velocity at time t = 2s, and (b) the acceleration a.

Solution. To find the velocity function $\upsilon(t)$ and acceleration we use Eqs. 2.10 and 2.13.

a) To get the function $\upsilon(t)$, we differentiate the position function $x(t)$ with respect to time (Eq. 2.10):

$$\upsilon = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(5 + 10t + 3t^2) = 10 + 6 \cdot t.$$

Setting t = 2 s in the above equation gives

$$\upsilon = 10 \text{ m/s} + 6 \text{ m/s} \cdot 2\text{s} = 22 \text{ m/s}.$$

b) To get the acceleration a, we differentiate the velocity function $\upsilon(t)$ with respect to time (Eq. 2.13): $a = \frac{\mathrm{d}\upsilon}{\mathrm{d}t} = 6 \text{ m/s}^2.$

The particle's velocity at time t = 2 s is 22 m/s and acceleration is 6 m/s^2.

***Example 2.10*. Acceleration at given *x*(*t*).**

A particle is moving in a straight line so that its position is given by the relation $x = 5 + 4t^2$. Calculate (a) its average acceleration during the time interval from t_1 = 2 s to t_2 = 5 s, and (b) its instantaneous acceleration as a function of time.

Solution. To determine acceleration, we first must find the velocity at t_1 and t_2 by differentiating $x(t)$ (Eq. 2.10). Then we use Eq. 2.11 to find the average acceleration, and Eq. 2.13 to find the instantaneous acceleration.

a) The velocity at any time t is

$$\upsilon = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(5 + 4t^2) = 8t.$$

Therefore, at t_1 = 2 s, υ_1 = 8·2 = 16 m/s and at t_2 = 5 s, υ_2 = 40 m/s. Thus, we obtain

$$a_{\text{av}} = \frac{\upsilon_2 - \upsilon_1}{\Delta t} = \frac{40 \text{ m/s} - 16 \text{ m/s}}{3 \text{ s}} = 8 \text{ m/s}^2.$$

b) The instantaneous acceleration at any time is

$$a = \frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(8t) = 8 \text{ m/s}^2.$$

The acceleration in this case is constant.

***Example 2.11*. Uniformly accelerated linear motion.**

A car starts from rest and then it is moving with a constant acceleration a = 2 m/s^2 during a 100 m race. How fast is the car going at the finish line?

Solution. We use equation (2.20), where we set $\upsilon_0 = 0$. Setting $\upsilon_0 = 0$ in Eq. 2.20 and solving it for t we obtain

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2(100\ \text{m})}{2\ \text{m/s}^2}} = 10\ \text{s}.$$

It takes a car 10 s to get a finish line.

2.5. FREELY FALLING OBJECTS

An important example of uniformly accelerated linear motion is that of an object falling freely near Earth's surface (fig. 2.13).

Galileo Galilei was the first to postulate that at a given location on the Earth and in the absence of air resistance, *all objects fall with the same constant acceleration.* This acceleration is called *the acceleration due to gravity (the free-fall acceleration)* on the surface of the Earth, and it is denoted by symbol g. Its magnitude is approximately $\boldsymbol{g} = \mathbf{9.80\ m/s^2}$ (at the surface of Earth). *Acceleration due to gravity is a vector as is any acceleration and its direction is downward, toward the center of the Earth* (fig. 2.13). It is independent on the body characteristics, such as mass, density, or shape; it is the same for all objects.

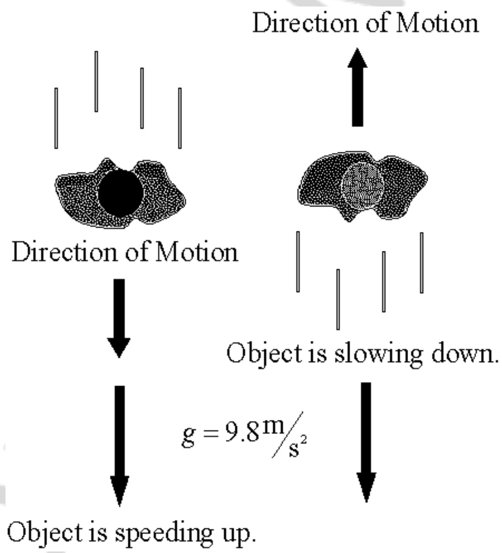


Fig. 2.13. The acceleration due to gravity $\boldsymbol{g}$

When dealing with freely falling bodies Eqs. 2.18–2.23 also describe this motion, where we replace $\boldsymbol{a}$ with the value of $\boldsymbol{g}$ given above. Also, since the motion is vertical, we refer the motion to the vertical coordinate $\boldsymbol{y}$ axis. As a rule, the direction of $\boldsymbol{y}$ axis coincides with the direction of the body motion. We take $\boldsymbol{y_0} = \mathbf{0}$ unless otherwise specified. Then the equations of the freely falling body motion (at $\boldsymbol{y_0} = \mathbf{0}$) are

$$\boldsymbol{\upsilon_y = \upsilon_{0y} + g_y t,} \tag{2.24}$$

$$\boldsymbol{s = \upsilon_0 t + \frac{g_y t^2}{2},} \tag{2.25}$$

$$\boldsymbol{\upsilon^2 = \upsilon_0^2 + 2 g_y s.} \tag{2.26}$$

NOTE. *In the above equations the projections of vectors* $\boldsymbol{\upsilon}$ *,* $\boldsymbol{\upsilon_0}$ *and* $\boldsymbol{g}$ *are positive if their directions coincide with the axis* ***OY*** *and they are negative in the opposite case*.

In the special case of a body thrown upward with an initial velocity $\boldsymbol{\upsilon_0}$, its acceleration is equal to the acceleration due to gravity $\boldsymbol{g}$ (in the absence of air resistance). As the body rises, its velocity decreases until it reaches the highest point (retarded motion), where its velocity is zero for an instant; then it descends, with the increasing velocity.

***Example 2.12*. Falling from a tower.**

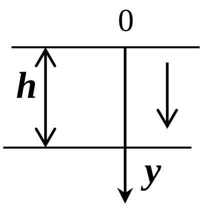


Fig. 2.14. Example 2.12

Suppose that a ball is dropped from a tower 70.0 m high. Calculate (a) how far will it have fallen after time interval t = 2.0 s? (b) how much time it takes for the ball to reach ground. Ignore air resistance. Assume that the initial velocity is equal to zero. (g = 9.80 m/s^2).

Solution. Let us take y as positive downward, so the acceleration due to gravity g is positive: g = +9.8 m/s^2 (fig. 2.14). For solving a problem we use Eq. 2.25, where we set $\upsilon_0 = 0$.

a) We set t = 2.0 s in Eq. 2.25:

$$s=\frac{gt^2}{2}=\frac{(9.8\text{ m/s}^2)(2s)^2}{2}=19.6\text{ m}.$$

The ball has fallen a distance of 19.6 m during the time interval t = 2 s.

b) Solving Eq. 2.25 for t and setting $s = h$ = 70 m gives

$$t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2\cdot 70\text{ m}}{9.8\text{ m/s}^2}}=\sqrt{14.3}\text{ s}=3.8\text{ s}.$$

It takes 14.3 ***s*** for the ball to reach ground.

***Example 2.13*. Ball thrown upward.**

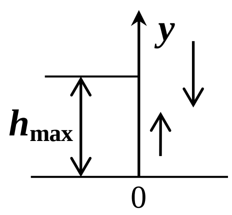


Fig. 2.15. Example 2.13

A person throws a ball upward into the air with an initial velocity of 10.0 m/s (fig. 2.15). Calculate (a) how maximum high it goes; (b) how much time it takes for the ball to reach maximum height. Ignore air resistance. (g = 9.80 m/s^2).

Solution. Let us choose y to be positive in the upward direction. The acceleration due to gravity is downward and so it has negative sign ($g_y = -g$ = –9.80 m/s^2) (fig. 2.15). For solving a problem we use Eqs. 2.24 and 2.26:

$$\upsilon = \upsilon_0 - gt, \quad \upsilon^2 = \upsilon_0^2 - 2gs.$$

We consider the time interval from when the ball leaves the thrower's hand until the ball reaches the highest point. At time t = 0 we have y_0 = 0, υ_0 = 10.0 m/s. As the ball rises, its speed decreases until it reaches the highest point. At time t_h (maximum height), the velocity of the ball is equal to zero.

a) To determine the maximum height h_{max}, we set $\upsilon = 0$ in Eq. 2.26 and solve it for $s = h_{max}$:

$$h_{max}=\frac{\upsilon_0^2-\upsilon^2}{2g}=\frac{\upsilon_0^2}{2g}=\frac{(10\text{ m/s})^2}{2(9.8\text{ m/s}^2)}=5.1\text{ m}.$$

The ball reaches a maximum height of 5.1 m.

b) The time t_h required for the ball to reach its highest point h_{max} we can calculate from the Eq. 2.24, where we set $\upsilon = 0$. Then

$$0 = \upsilon_0 - gt_h,$$

$$t_h = \frac{\upsilon_0}{g} = \frac{10.0\ \text{m/s}}{9.8\ \text{m/s}^2} = 1.02\ \text{s}.$$

The time required for the ball to reach the maximum height is 1.02 s.

2.6. UNIFORM CIRCULAR MOTION

Uniform circular motion occurs when an object (point particle) moves in a circular path at constant speed.

In a uniform circular motion, a particle covers equal distances within equal interval of time, but the direction of motion changes at every point as shown in fig. 2.16. In this case the velocity of the body is referred to as *linear velocity*. A circular motion is an example of an accelerated motion with a constant speed.

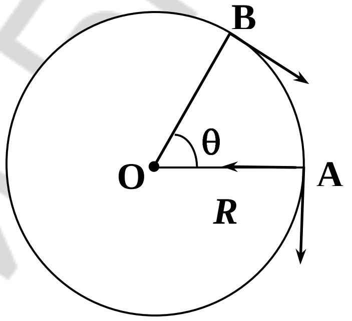


Fig. 2.16. A linear velocity of a particle rotating about an axis O

The angular displacement of a particle is measured by the angle θ covered by the circle radius ***R*** and it is subtended at the center of the circle (fig. 2.16).The fixed axis around which motion takes place is called ***axis of rotation*** (point O in fig. 2.16). In a uniform circular motion, a particle undergoes the same angular displacement during the same time interval.

The angular displacement of a particle can be given in degrees or in radians.

One radian is defined as the angle subtended at the center of a circle by an arc with the length equal to the radius of the circle (fig. 2.17). If θ is the angle subtended by an arc ***AB*** of length ***l*** at the center of a circle of radius ***R***, then

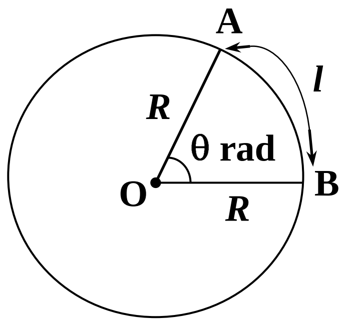


Fig. 2.17. A radian definition

$$\boldsymbol{\theta\ (\text{radian}) = \frac{l}{R}}. \tag{2.27}$$

In one complete rotation or 360°, we have

$$\boldsymbol{\frac{l}{R} = \frac{2\pi R}{R} = 2\pi\ \text{radian}.}$$

$$\boldsymbol{1\ \text{radian} = \frac{180^\circ}{\pi} = 57^\circ 18'.}$$

Angular velocity is the angle described by a particle during a unit time. Angular velocity is represented by Greek letter ω (Omega) and is defined as

$$\boldsymbol{\omega = \frac{\theta}{t}}. \tag{2.28}$$

The units of angular velocity ω are radian per second (rad/s).

The relation between the linear υ and angular ω velocities is given by the following expression:

$$\boldsymbol{\upsilon = R \cdot \omega}, \tag{2.29}$$

where $\boldsymbol{R}$ is the circle radius.

Change of the linear velocity υ direction is described by the ***centripetal acceleration*** $\boldsymbol{a_c}$. Its direction is along the radius $\boldsymbol{R}$ to the circle center (fig. 2.16). The magnitude of the centripetal acceleration is defined as

$$\boldsymbol{a_c = \frac{\upsilon^2}{R}}, \tag{2.30}$$

where $\boldsymbol{R}$ is the circle radius.

Substituting (2.29) in this equation we obtain the another equation for the centripetal acceleration:

$$\boldsymbol{a_c = \omega^2 R}. \tag{2.31}$$

Time required to complete one rotation is called the time period and represented by $\boldsymbol{T}$. The circle length is equal to $\boldsymbol{2\pi R}$, thus time period is given as

$$\boldsymbol{T = \frac{2\pi R}{\upsilon} = \frac{2\pi}{\omega}} \tag{2.32}$$

$$\text{or } \boldsymbol{\omega = \frac{2\pi}{T}}. \tag{2.33}$$

The number of rotations made by the particle in 1 second is called frequency of rotation. It is represented by Greek letter ν (nu):

$$\boldsymbol{\nu = \frac{1}{T}}. \tag{2.34}$$

The SI units for frequency of rotation are hertz (Hz).

The angular velocity is related to the frequency of rotation by the following equation:

$$\boldsymbol{\omega = 2\pi\nu}. \tag{2.35}$$

From the equations 2.30–2.35 the following equations for the centripetal acceleration can be derived:

$$\boldsymbol{a_c = \frac{\upsilon^2}{R} = \omega^2 R = \frac{4\pi^2 R}{T^2} = 4\pi^2\nu^2 R}. \tag{2.36}$$

Example 2.14.

The platter of the hard drive of a computer rotates at 6000 rev/min (revolutions per min). (a) What is the angular velocity of the platter? (b) If the reading head of the drive is located 3.5 cm from the rotation axis, what is the linear speed of the point on the platter just below it?

Solution. **a)** To find the angular velocity we use Eq. 2.35, where we set frequency of rotation ν = 6000 rev/min = 6000 rev/60 s = 100 Hz. Then the angular velocity is $\omega = 2\pi\nu = 2\cdot(3.14\text{ rad})\cdot(100\text{ Hz}) = 628$ rad/s.

b) The linear velocity of the point a 3.5 cm out of the rotation axis is given by the Eq. 2.30, where we set $R = 3.5\text{ cm} = 3.5 \cdot 10^{-2}$ m:

$$\upsilon = R\cdot\omega = (3.5 \cdot 10^{-2}\text{ m}) \cdot (628\text{ rad/s}) = 22\text{ m/s}.$$

PROBLEMS

1. A train 50 meter long passes a bridge 250 m long at the speed of 9 km/h. How long will it take to completely pass over the bridge? (Answer: 2 min)

2. A ball rolling at the speed of 1 m/s was stopped within 1 meter. What was the average retardation applied to ball? How long did it take to stop it? (Answer: 0.5 m/s^2, 2 s)

3. A motor car starts from rest and accelerates uniformly for 30 s to a speed of 72 km/hr. It then moves with uniform velocity and is finally brought to rest in 50 meter with a constant retardation. If the total distance travelled is 950 meter, find the acceleration, retardation and the total time taken. (Answer: 0.67 m/s^2, 4 m/s^2, 65 s)

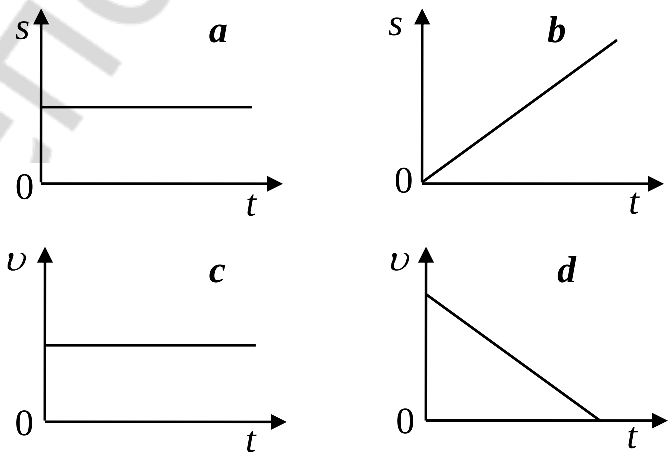
4. A car starts from rest and moves with uniform acceleration. Its velocity is 25 cm/s after 5 s and 34 cm/s after 8 s. Calculate the distance that it will travel in tenth second. (Answer: 28.5 cm)

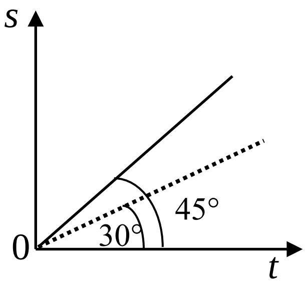
5. A stone is thrown upwards from the surface of earth with initial speed of 5 m/s. What is the maximum height reached by a stone? (Answer: g = 10 m/s^2; 1.25 m/s)

6. What is the angular speed of (a) the second (b) the minute and (c) the hour hand of an analog watch? (Answer: 0.105 rad/s, $1.75\cdot10^{-3}$ rad/s, $1.45\cdot10^{-4}$ rad/s)

TESTS (Sections 2.1–2.5)

1. Which of following graphs represents motion with uniform speed?





2. Two straight lines drawn at the same displacement time graph make angles 30° and 45° with *X*-axis as shown The ratio of two velocities is:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{\sqrt{3}}$ c) $\frac{1}{3}$ d) $\sqrt{2}$

3. A body covers half of its distance with speed "u" and other half with the speed "υ", the average speed of the body is:

a) $\frac{u+\upsilon}{2}$ b) $\frac{u-\upsilon}{2}$ c) $\frac{2u\upsilon}{u+\upsilon}$ d) $\frac{u+\upsilon}{2u\upsilon}$

4. A body goes from A to B with a velocity 40 km/hr and comes back from B to A with a velocity of 50 km/hr, the average velocity during the whole journey is:

a) 45 km/hr b) 44.4 km/hr c) zero km/hr d) 48 km/hr

5. At first half of the distance a body moves with a velocity 40 m/s and at second half — with a velocity 50 m/s at same direction, so the average velocity during its whole journey is:

a) 45 m/s b) zero m/s c) 44.44 m/s d) 48 m/s

6. Which of the following statements is not true?

a) velocity, acceleration and displacement are vectors;

b) a vector quantity has only magnitude while scalar has both magnitude and direction;

c) mass, work, energy, moment of inertia are scalars.

7. Acceleration is the rate of change of

a) speed b) position c) velocity d) distance

8. A particle is moving with a velocity υ_1 m/s and after $t(s)$ the velocity changes to υ_2 m/s. The average acceleration is:

a) $\frac{\upsilon_1+\upsilon_1}{t}$ m/s^2 b) $\frac{\upsilon_2-\upsilon_1}{t}$ m/s^2

c) $(\upsilon_1+\upsilon_2)\cdot t$ m/s^2 d) none of the above

9. An object starting from rest covers distances in direct proportion to the square of time. Its acceleration is:

a) increasing b) constant

c) zero d) none of the above

10. The displacement *x* of *a* particle along a straight line at a time *t* is $x = a_0 + a_1 t + a_2 t^2$. The acceleration of the particle is:

a) a_0 b) a_1 c) $2a_2$ d) a_2

11. If the speed of a car increases by 3 times, how does the distance needed to stop it change:

a) 3 times b) 6 times c) 9 times d) None of the above

12. A body starts from rest and moves with uniform acceleration "a". The distance covered during the nth second will be:

a) $a(n-1)$ b) $a(n-\frac{1}{2})$ c) $a(2n-1)$ d) $a(2n+1)$

13. A body starting from rest travels 150 m in 8th second. Its uniform acceleration is:

a) 15 m/s^2 b) 10 m/s^2 c) 20 m/s^2 d) 30 m/s^2

14. An object is projected upwards with a velocity of 100 m/s. It strikes the ground back in (g = 10 m/s^2):

a) 10 s b) 20 s c) 5 s d) 15 s

15. A ball thrown vertically upwards with an initial velocity of 19.6 m/s returns to thrower in 4 s. The maximum height reached by it is:

a) 9.8 m b) 44.1 m c) 19.6 m d) 26.7 m

16. Two bodies of mass m_1 and m_2 are dropped from rest from heights h_1 and h_2. The ratio of their times to reach the ground are:

a) $h_1 : h_2$ b) $\sqrt{h_1} : \sqrt{h_2}$

c) $m_1\sqrt{h_1} : m_2\sqrt{h_2}$ d) $m_1h_1 : m_2h_2$

Tests (Section 2.6)

1. A body is said to be in uniform circular motion when its linear velocity:

a) remains constant both in magnitude and direction;
b) remains constant in magnitude but changes in direction;
c) remains constant in direction but changes in magnitude;
d) none of the above.

2. The acceleration of a body performing uniform circular motion with linear speed υ in circle of radius R is:

a) $\frac{\upsilon^2}{R}$ b) $\frac{\upsilon}{R}$ c) $\frac{\upsilon}{R^2}$ d) $\upsilon^2 \cdot R$

3. A car going round a circular path at constant speed

a) has a constant linear velocity;
b) has a constant acceleration;
c) has a constant momentum;
d) none of the above.

4. A body moves in a circle of radius R. After completing the circular path once it returns at the point of start. The displacement of the body is:

a) $2\pi R$ b) πR c) zero d) πR^2

5. A body moving in a circle of radius R with constant speed makes n revolutions per second. Its centripetal acceleration is:

a) $2\pi nR$ b) $4\pi^2 n^2 R$ c) $\pi n^2 R$ d) $\pi^2 n^2 R$

6. Two bodies of masses m_1 and m_2 are moving in concentric circles of radii r_1 and r_2 such that the frequencies of revolutions are the same. The ratio of centripetal accelerations is:

a) $R_1^2 : R_2^2$ b) $R_1 : R_2$ c) $\sqrt{R_1} : \sqrt{R_2}$ d) $R_2 : R_1$

7. In the above problem the ratio of their angular velocities is:

a) $R_1 : R_2$ b) $R_1^2 : R_2^2$ c) 1 : 1 d) $R_2^2 : R_1^2$

8. The ratio of the angular speeds of minute hand and hour hand of a clock is:

a) 6 : 1 b) 12 : 1 c) 1 : 6 d) none of the above

9. In uniform circular motion which is true?

a) both velocity and acceleration are constant;
b) both acceleration and speed are not constant;
c) both acceleration and speed are constant;
d) both velocity and acceleration are not constant.

10. A particle moves in a plane with a constant speed with its direction changing. The path of the particle is:

a) straight line;
b) an arc of the circle;
c) a parabola;
d) an ellipse.

11. The length of second's hand in a watch is 1 cm. The change in speed of its tip in 15 seconds is:

a) Zero b) $\frac{\pi}{30}$ cm/s c) $\frac{\pi}{30\sqrt{2}}$ cm/s d) $\frac{\pi}{30}\sqrt{2}$ cm/s

3. DYNAMICS

In this Chapter we consider what makes the objects to move. The connection between the force and motion studies the subject called ***dynamics***.

3.1. NEWTON'S LAWS OF MOTION

Before discussion Newton's laws of motion, let us give definitions of some useful terms.

Inertia. Inertia is the property of a body to maintain its state of rest or uniform motion in a straight line.

Mass. The mass is a measure of the inertia of an object. It is the characteristic of a body that relates the body's acceleration to the net force causing the acceleration. The more mass a body has, the greater the force needed to impart it a particular acceleration. It is harder to start it moving from rest, or to stop it when it is moving. Mass (**m**) is a scalar quantity. The SI unit for mass is kilogram (**kg**).

Force. Force is a push or a pull which changes or tends to change the state of rest or of uniform motion of a body in a straight line. We need a force to overcome inertia of a body. When you push a body, you are exerting a force on it (fig. 3.1). We often call it ***contact force*** because the force is exerted when one object comes in contact with another object. On the other hand, we say that an object falls because of ***the force of gravity***.

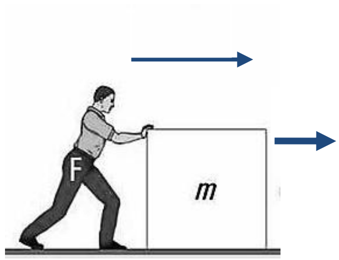


Fig. 3.1. A force exerted on a body by a person

Force ($\boldsymbol{F}$) is a vector and has both direction and magnitude (fig. 3.1). We can represent any force on a diagram by an arrow. The direction of the arrow is the direction of push or pull, and its length is drawn proportional to the magnitude of force. The SI unit for force is newton (N).

If two or more forces act on a body, we find the net (resultant) force ($\boldsymbol{R}$) by adding them as vectors (see example in fig. 3.2).

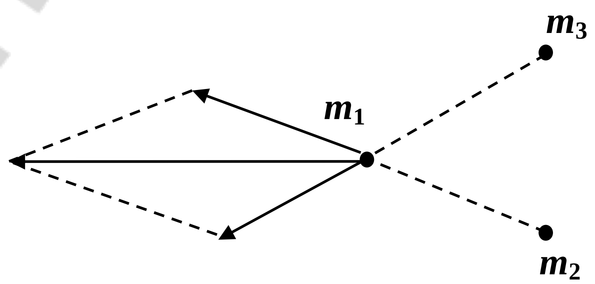


Fig. 3.2. The net force $\boldsymbol{R}$ is a vector sum of $\boldsymbol{F_{12}}$ and $\boldsymbol{F_{13}}$ forces ($\boldsymbol{R} = \boldsymbol{F_{12}}+\boldsymbol{F_{13}}$)

A single force that has the same magnitude and direction as the calculated net force would then have the same effect as all the individual forces. This fact is called ***the principle of superposition for forces***.

Newton's first law of motion (law of inertia): *every body continues in its state of rest or of uniform motion in a straight line, as long as no net force acts on it. This is called principle of inertia.*

From the Newton's first law it is evident that a body by itself is incapable of changing its state of rest or of uniform motion in a straight line. This incapability is known as inertia. Hence first law is called law of inertia.

The reference frames in which Newton's first law does hold are called ***inertial reference frames*** or ***inertial frames***. Usually approximation is made that a reference frame fixed on the Earth is an inertial frame. Any reference frame that moves with constant velocity relative to an inertial frame is also inertial reference frame.

Reference frames in which the law of inertia is not valid, such as accelerating reference frames, are called ***noninertial reference frames*** or ***noninertial frames***.

Newton's second law of motion: *acceleration of a body is directly proportional to the net force acting on it, and is inversely proportional to the body's mass.* The direction of the acceleration is in the direction of the net force acting on the body.

$$\vec{a} = \frac{\vec{F}}{m} \text{ or } \vec{F} = m\vec{a}, \qquad (3.1)$$

where $\boldsymbol{a}$ is the acceleration, $\boldsymbol{m}$ *is* the mass, and $\boldsymbol{F}$ is the *net force acting* on the body.

When n forces different in magnitude and direction act of the body, $\boldsymbol{F}$ means the *vector sum of all the forces* acting on the object, which we defined above as the net (resultant) force. Then

$$\sum_{i=1}^{n} \vec{F_i} = m\vec{a}, \qquad (3.2)$$

where $\sum_{i=1}^{n} \vec{F_i} = \vec{F_1} + \vec{F_2} + \vec{F_3} + ... + \vec{F_n}$.

Only external forces, i. e. forces exerted on **the body** by other bodies, are to be included.

Fig. 3.3. The action and the reaction forces exerted by bodies

Newton's third law of motion: *whenever one object exerts a force on a second object, the second exerts an equal force in the opposite direction on the first.* Forces come in pairs (fig. 3.3).

Commonly one of these forces is called ***the action force***. The other one is then called ***reaction force***. The term ***action*** means the force exerted on the first body by second body ($\boldsymbol{F_{12}}$), while ***reaction*** means force exerted on the second body by the first body ($\boldsymbol{F_{21}}$). Then

$$\overrightarrow{\boldsymbol{F_{12}}} = -\overrightarrow{\boldsymbol{F_{21}}} \text{ or } \left|\overrightarrow{\boldsymbol{F_{12}}}\right| = \left|\overrightarrow{\boldsymbol{F_{21}}}\right|. \qquad (3.3)$$

Third law of motion can be formulated as "to every action there is an equal and opposite reaction". But it is very important to remember that the "action" and "reaction" forces are acting on different objects.

3.2. MAIN FORCES

3.2.1. THE GRAVITATIONAL FORCE

Galileo Galilei claimed that all objects dropped near the surface of the Earth fall with the same acceleration, ***g***, if air resistance is negligible. The force that causes this acceleration of mass ***m*** is the ***gravitational force*** $\boldsymbol{F_g}$, that can be written as

$$\boldsymbol{F_g = mg.} \qquad (3.4)$$

This force is directed down toward the center of the Earth (fig. 3.4). The *gravitational force (force of gravity)* is explained by ***Newton's law of universal gravitation*** that states: *every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them*. This force acts along the line joining the two particles (fig. 3.5). *The magnitude of the gravitational force* can be written as

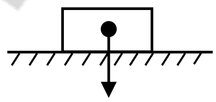


Fig. 3.4. The gravitational force

$$\boldsymbol{F_g = G{\cdot}m_1m_2/r^2,} \qquad (3.5)$$

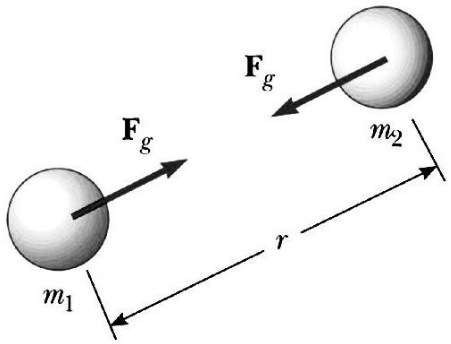


Fig. 3.5. Newton's law of universal gravitation

where $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are the masses of the two particles, ***r*** is the distance between them, and ***G*** is a universal constant which has the same numerical value for all objects ($\boldsymbol{G = 6.67{\cdot}10^{-11}}$ **N·m²/kg²**).

3.2.2. GRAVITY NEAR THE EARTH SURFACE

When Eq. 3.5 is applied to the gravitational force between the Earth and an object at its surface ($\boldsymbol{F_g}$), $\boldsymbol{m_1}$ becomes the mass of the Earth ***M***, $\boldsymbol{m_2}$ becomes the mass of the object ***m***, and ***r*** becomes the distance of the object from the Earth's center, which is approximately the radius of the Earth ***R***. Thus,

$$\boldsymbol{F_g = G{\cdot}mM/R^2.} \qquad (3.6)$$

Combining Eq. (3.4) and (3.6) we obtain

$$\boldsymbol{mg = G{\cdot}mM/R^2.} \qquad (3.7)$$

Solving Eq. 3.7 for g gives the acceleration of gravity at the Earth's surface:

$$\boldsymbol{g} = \frac{\boldsymbol{G \cdot M}}{\boldsymbol{R}^2} \approx \mathbf{9.8\ m/s^2}. \tag{3.8}$$

Thus, the acceleration of gravity at the surface of the Earth, $\boldsymbol{g}$, is determined by the Earth mass $\boldsymbol{M}$ and the radius of the Earth $\boldsymbol{R}$.

3.2.3. THE FORCE OF ELASTICITY AND HOOKE'S LAW

The elasticity forces are revealed in a response of a body to the action of the external forces. The external force is called the load.

Changes of shape and (or) dimensions of the sample which is made from some material under the action of the external forces is called **strain (deformation)**.

Elasticity is the property of a body to preserve its shape and dimensions after load removing.

For example, let us consider the case of *an axial tension (or compression)* of a body (fig. 3.6). A load tends to stretch (elongate) a body (fig. 3.6, *a*). Then absolute deformation (the change in length) is equal to $\boldsymbol{x = l - l_0}$, where $\boldsymbol{l_0}$ is the original length, $\boldsymbol{l}$ is the elongated length. Relative deformation (strain) $\boldsymbol{\varepsilon}$ is defined as:

$$\boldsymbol{\varepsilon} = \frac{\boldsymbol{x}}{\boldsymbol{l_0}}. \tag{3.9}$$

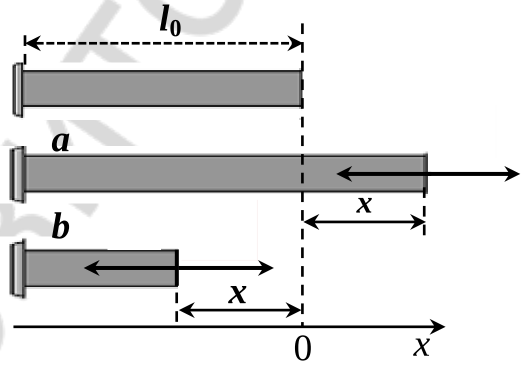


Fig. 3.6. The force of elasticity $\boldsymbol{F_{el}}$ upon tension (*a*) and compression (*b*)

The value of ε is dimensionless, it is usually expressed in percents (%).

When an external force F acts upon a solid body (the body is deformed), a reaction force (internal force) arises within the body that is equal in magnitude but opposite in direction to the external force (fig. 3.6). This is a **force of elasticity** $\boldsymbol{F_{el}}$ and it is sometimes called a "restoring force". According to *Hooke's law* a force of elasticity is directly proportional to deformation:

$$\boldsymbol{F_{el} = -k \cdot x}, \tag{3.10}$$

where $\boldsymbol{k}$ is the **stiffness constant**, $\boldsymbol{x}$ is an absolute deformation.

The internal force is characterized by ***the mechanical stress* σ**. For an axial tension (or compression) the value of ***stress* σ** (**tensile** or **compressive stress**) is defined as the average force F per unit cross-sectional area S within the body on which external force acts:

$$\sigma = \frac{F}{S}. \qquad (3.11)$$

In the SI system stress is measured in the pascals (symbol Pa), which is defined as one newton per square meter:

$$\sigma = \frac{N}{m^2} = Pa.$$

In terms of strain and stress *Hooke's law* can be defined as:

$$\sigma = E \cdot \varepsilon, \qquad (3.12)$$

where E is called ***Young's modulus or the modulus of elasticity***.

Thus *Hooke's law states that the stress is directly proportional to the strain upon elastic deformation.*

Substituting Eq. 3.9 and 3.11 for the strain and stress, respectively, in the Eq. 3.12 we obtain:

$$\frac{F}{S} = \frac{E \cdot x}{l_0}. \qquad (3.13)$$

Then $F = \frac{E \cdot S}{l_0} x$, and from Eq. 3.13 it follows that

$$k = \frac{E \cdot S}{l_0}. \qquad (3.14)$$

Thus the stiffness constant k is directly proportional to the modulus of elasticity, and it is dependent on the body shape and dimensions.

3.2.4. NORMAL FORCE

For an object resting on a table, the table exerts upward force (N), but it remains stationary (fig. 3.7). The reason is that the table is compressed slightly beneath the object, and due to its elasticity, it pushes up on the object as shown.

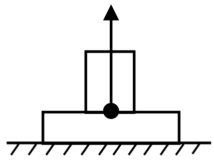


Fig. 3.7. The normal force

The force exerted by the table is often called a contact force, since it occurs when two objects are in contact. The push on an object from the table is a ***normal force*** (N) (sometimes it is denoted by symbol F_N). The name comes from the mathematical term *normal*, meaning perpendicular: the force on an object from the table is perpendicular to the table (fig. 3.7). Thus when a body presses against a surface, the surface deforms and pushes on the body with a normal force that is perpendicular to the surface.

3.2.5. TENSION

When a cord (or a rope or other such object) is attached to a body and pulled taut, the cord pulls on the body with a force directed away from the body and along the cord (fig. 3.8). The force is often called ***a tension force*** (***T***) because the cord is said to be in a state of tension (or to be under tension), which means that it is being pulled taut. The tension in the cord is the magnitude ***T*** of the force on the body.

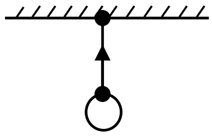
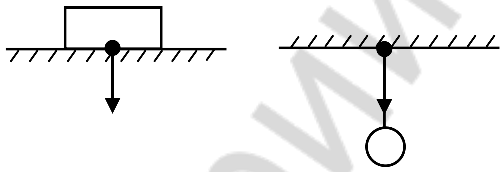


Fig. 3.8. The tension force

A cord is often said to be massless (meaning its mass is negligible compared to the body's mass) and unstretchable.

3.2.6. WEIGHT

Weight of the body (***W*** or ***P***) is the force with which the body acts on a support or on a cord. The weight force is applied not to the body but to the support or to the cord (fig. 3.9).



a ***b***

Fig. 3.9. Weight of the body ***W***

In accordance with the Newton's third law the weight force ***W*** acting on a support (or on a cord) and the normal force ***N*** (or tension force ***T***) exerted by a support on the body (reaction force) are equal in magnitude (***W*** **=** ***N*** and ***W*** **=** ***T***), and oppositely directed (fig. 3.10):

$$\vec{W} = -\vec{N} \text{ or } \vec{W} = -\vec{T}. \quad (3.15)$$

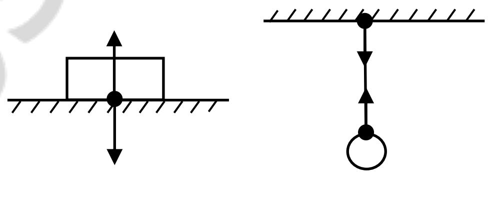


Fig. 3.10. Weight of the body ***W*** and the reaction forces (normal ***N*** and tension ***T*** forces)

3.2.7. FRICTION

If we either slide or attempt to slide a body over a surface, the motion is resisted by a bonding between the body and the surface. The resistance is considered to be a single force called either ***the frictional force*** **(**$\boldsymbol{F_{fr}}$**)** or simply ***friction***.

We focus our attention on sliding friction, which is usually called ***kinetic friction*** when object slides across a surface. This force is directed along the surface, opposite to the direction of the motion (fig. 3.11).

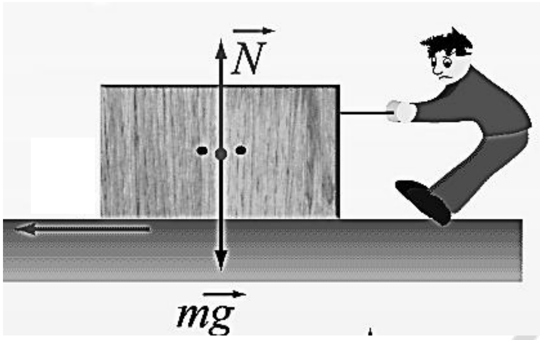


Fig. 3.11. The friction force $\boldsymbol{F_{fr}}$

Sometimes, to simplify a situation, friction is assumed to be negligible (the surface, or even the body, is said to be frictionless).

The magnitude a frictional kinetic force is given by

$$\boldsymbol{F_{fr} = \mu_k \cdot N,} \tag{3.16}$$

where $\boldsymbol{\mu_k}$ is the coefficient of kinetic friction, $\boldsymbol{N}$ is the magnitude of the normal force. The coefficient $\boldsymbol{\mu_k}$ is dimensionless and its value depends on the nature of the two surfaces in contact.

There is also static friction, which refers to a force parallel to the two surfaces that can arise even when they are not sliding. The force of static friction depends on various factors and it may change from zero to its maximum value given by $\boldsymbol{(F_{fr})_{max} = \mu_s \cdot N}$, where μ_s is the coefficient of static friction generally being greater than μ_k. However when solving problems it is commonly suggested that both of these coefficients are equal: $\boldsymbol{\mu_s = \mu_k}$.

The *static friction force* is directed along the surface, opposite to the direction of the intended motion.

3.2.8. Problem solving

When solving problems involving Newton's laws and forces, it is very important to draw a diagram showing all the forces acting on each object involved. Such a diagram is called a *free-body diagram* (fig. 3.12), or ***force diagram***: choose one object, and draw an arrow to represent each force acting on it. Include every force acting on that object. Do not show forces that the chosen object exerts on other objects. Only forces acting on a given object can be included in the Newton's second law equation (Eq. 3.1). If your problem involves more than one object, a separate free-body diagram is needed for each object. For the case shown in fig. 3.12, the forces that could be acting are gravity and contact forces (one object pushing or pulling another, normal force, friction).

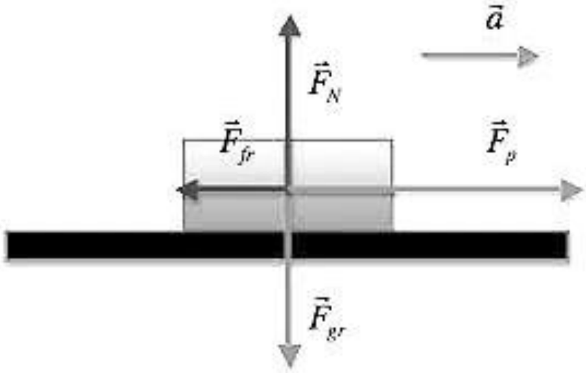


Fig. 3.12. A free-body diagram

Newton's second law involves vectors, and it is usually necessary to resolve vectors into components. Choose *x* and *y* axis in a way that simplifies the calculation. Then for each object apply Newton's second law to the $\boldsymbol{x}$ and $\boldsymbol{y}$ components separately (i. e. $\sum \boldsymbol{F_x} = \boldsymbol{m a_x}$, $\sum \boldsymbol{F_y} = \boldsymbol{m a_y}$).

NOTE. *The weight of the body is always applied to the support (or a cord), and not to the body, this is why it is not included in the Newton's second law (force* $\boldsymbol{P}$ *in fig. 3.13). To find the weight of the body, it is necessary to solve Newton's second law equation, determine the magnitude of the reaction force (**N** or **T**) which is equal to the weight **W**, according to the Eq. 3.15.*

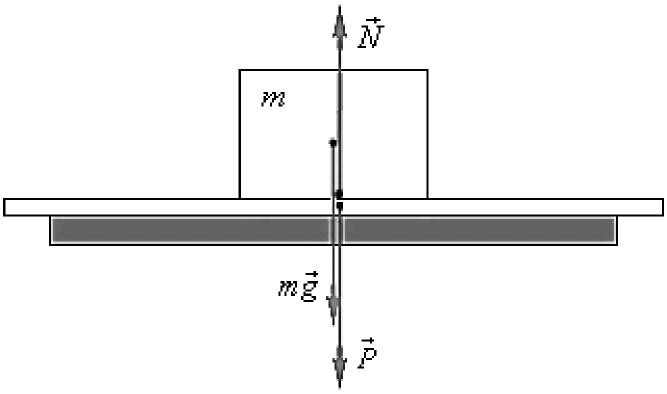


Fig. 3.13. A box resting on a table

Let us determine the weight of a box ($\boldsymbol{P}$) resting on the smooth (frictionless) horizontal surface of a table (fig. 3.13). The forces acting on the body are the normal force ($\boldsymbol{N}$) and the gravitational force ($\boldsymbol{F_g} = \boldsymbol{mg}$) as shown in fig. 3.13. The box is at rest, so its acceleration $\boldsymbol{a} = \boldsymbol{0}$ and in accordance with the Newton's second law the net force acting on it must be zero. Hence

$$\overrightarrow{\boldsymbol{N}} + \boldsymbol{m}\overrightarrow{\boldsymbol{g}} = \boldsymbol{0} \text{ or } \overrightarrow{\boldsymbol{N}} = -\boldsymbol{m}\overrightarrow{\boldsymbol{g}}.$$

Thus the magnitude of the normal force $\boldsymbol{N} = \boldsymbol{mg}$. In accordance with the Newton’s third law the magnitude of normal force on the box is equal to the box’s weight, so $\boldsymbol{W} = \boldsymbol{N} = \boldsymbol{mg}$. Thus weight of the body is equal to the magnitude of the gravitational force on the body just in case if it is located on a resting support (or on a resting cord) with respect to the Earth. If a support (or a cord) with a body accelerates down or upward, the weight $\boldsymbol{W} \neq \boldsymbol{mg}$.

***Example. 3.1.* Weight.**

A passenger of mass 70 kg descends in an elevator that accelerates at 2.5 m/s^2 downward. He stands on scales that shows in kg. (a) During this acceleration, what does the scales show? (b) What does the scales show when the elevator descends at a constant speed of 0.5 m/s?

Solution. We use Newton’s second law only in an inertial frame. If the cab accelerates, then it is *not* an inertial frame. So we choose the ground to be our inertial frame and make any measure of the passenger’s acceleration relative to it.

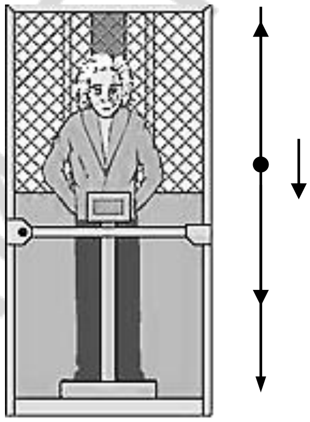


Fig. 3.14. Example 3.1. A free-body diagram for a passenger standing on the scales

The fig. 3.14 shows all the forces acting on the passenger. The direction of the acceleration is downward, so we choose the positive direction of y axis as down. From Newton’s second law it follows

$$F_g - N = ma.$$

a) The magnitude of the normal force (N) acting on the passenger is given by:

$$N = F_g - ma = mg - ma = m\cdot(g - a) = (70\text{ kg})\cdot(9.8\text{ m/s}^2 - 2.5\text{ m/s}^2) = 511\text{ N}.$$

So passenger’s weight is $W = N = 511$ N. The scales shows in kg

$$m = \frac{N}{g} = \frac{511\,N}{9.8\text{ m/s}^2} = 52.1\text{ kg}.$$

b) When the elevator descends at a constant speed, the acceleration is equal to zero ($a = 0$), so by Newton’s second law we obtain

$$F_g - N = 0 \text{ or } N = F_g = mg = (70\text{ kg})\cdot 9.8\ (\text{m/s}^2) = 686\text{ N}.$$

Thus passenger’s weight is W = N = 686 N and the scales shows his mass 70 kg.

***Example 3.2.* Newton’s second law.**

A box of mass m = 5 kg is pulled along a floor by a cord that exerts a force of magnitude F_P = 25 N at the angle 30°. The magnitude of the box’s acceleration is 2 m/s^2. What is the coefficient of kinetic friction?

Solution. The free-body diagram is shown in fig. 3.15. The forces on the box are the gravitational force $F_g = mg$, the normal force exerted by the floor F_{N}, the applied force F_P, and the friction force F_{fr}. We use Newton's second law given by Eq. 3.1.

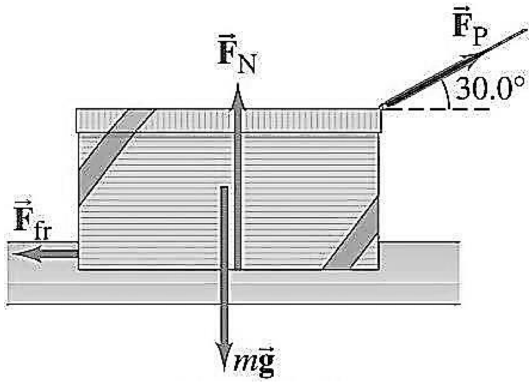


Fig. 3.15. Example 3.2. Free body diagram for a box pulled by a force of magnitude F_p at the angle 30°

Let us find projections of the forces and the acceleration vectors on the coordinate axes (x and y). We choose the upward direction as the positive y direction. In the vertical direction there is no motion (a_y = 0), so Newton's second law in the vertical direction gives $F_y = 0$, then

$$F_N + F_P \cdot \sin\alpha - mg = 0,$$

where the minus sign means that the gravitational force F_g acts in the negative y direction (m and g are magnitudes).

From this equation we obtain that the normal force is given by

$$F_N = mg - F_P \cdot \sin\alpha.$$

Now we apply Newton's second law for the horizontal (x) direction (positive to the right), and include the friction force:

$$F_P \cdot \cos\alpha - F_{fr} = ma.$$

Then the frictional force is given by

$$F_{fr} = \mu F_N = F_P \cdot \cos\alpha - ma.$$

Substituting in this equation expression for F_N obtained above gives the coefficient of kinetic friction (μ).

$$\mu = \frac{F_{fr}}{F_N} = \frac{F_P \cdot \cos\alpha - ma}{mg - F_P \cdot \sin\alpha} = \frac{25\ \text{N} \cdot \cos 30^\circ - (5\ \text{kg}) \cdot (2\text{m/s}^2)}{(5\ \text{kg}) \cdot (9.8\text{m/s}^2) - 25\ \text{N} \cdot \sin 30^\circ} = \frac{12.5\sqrt{3} - 10}{49 - 12.5} = 0.3.$$

The coefficient of kinetic friction μ = 0.3.

3.3. CONSERVATION OF MOMENTUM

3.3.1. LINEAR MOMENTUM AND IMPULSE EQUATION

Amount of motion present in body is called **linear momentum** (momentum) of the body. It is represented by a vector $\boldsymbol{p}$. Linear momentum of a body having mass $\boldsymbol{m}$ and moving at any instant of time with a velocity $\boldsymbol{\upsilon}$, is defined as

$$\boldsymbol{p} = \boldsymbol{m\upsilon}. \qquad (3.17)$$

The units for linear momentum in SI system are kg·m/s. The adjective linear is often dropped, but it serves to distinguish from angular momentum. Since mass $\boldsymbol{m}$ is always a positive scalar quantity, from Eq. 3.17 it follows that vectors $\boldsymbol{p}$ and $\boldsymbol{\upsilon}$ have the same direction.

In mechanics several bodies form ***system***. The forces can be of two types: ***external and internal forces***. Any force acting on the body system from the bodies outside the system is called ***an external force***. Forces between bodies inside the system are called ***internal forces***. Internal forces cannot accelerate the system. A system is called ***isolated*** if the resultant external force acting on a system of bodies is zero (or no external forces act), the only forces acting are those between objects of the system.

By substituting the expression for acceleration $\vec{\boldsymbol{a}} = \dfrac{\overrightarrow{\upsilon_2} - \overrightarrow{\upsilon_1}}{\boldsymbol{t}}$ into the second Newton's law equation (Eq. 3.1), we obtain

$$\boldsymbol{F} \cdot \boldsymbol{t} = \boldsymbol{m}\,\upsilon_2 - \boldsymbol{m}\,\upsilon_1 \tag{3.18}$$

$$\text{or } \boldsymbol{F} \cdot \boldsymbol{t} = \boldsymbol{p}_2 - \boldsymbol{p}_1, \tag{3.18a}$$

where υ_1 and $\boldsymbol{p}_1$ is the velocity and momentum of the body at some initial instant of time, υ_2 and $\boldsymbol{p}_2$ is the velocity and momentum of the body after certain time interval *t*.

The product of the force multiplied by the time interval during which it acts upon a body ($\boldsymbol{F} \cdot \boldsymbol{t}$) is called ***impulse of a force***.

A change of momentum of the body is equal to impulse of a force acting on a body (Eq. 3.18). This relation is called impulse equation.

3.3.2. THE LAW OF CONSERVATION OF MOMENTUM

Now consider, for example, an isolated system of two bodies. Suppose that the bodies of mass $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are moving with the velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and let them to interact with each other (fig. 3.16). Let the two bodies continue moving in the same direction with the velocities υ_1 and υ_2 after interaction.

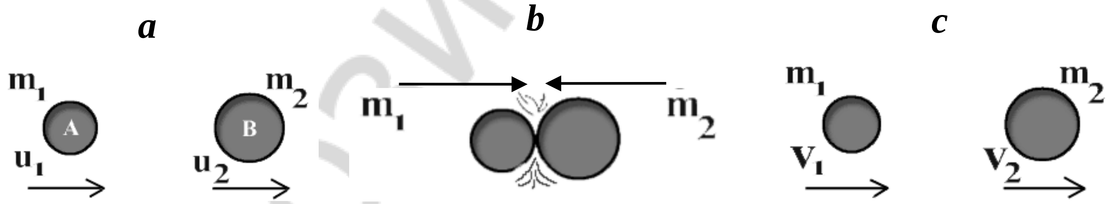


Fig. 3.16. Conservation of momentum in collision of two balls:
a — before collision; *b* — at collision; *c* — after collision

According to the Newton's third law of motion the forces acting on the bodies are equal in magnitude and oppositely directed, i. e. $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$

Change of momentum of the first body

$$\boldsymbol{F}_{12}\boldsymbol{t} = \boldsymbol{m}_1\,\upsilon_1 - \boldsymbol{m}_1\boldsymbol{u}_1 \tag{3.19}$$

Change of momentum of the second body

$$\boldsymbol{F}_{21}\boldsymbol{t} = \boldsymbol{m}_2\upsilon_2 - \boldsymbol{m}_2\boldsymbol{u}_2, \tag{3.20}$$

where ***t*** is the time of acting of a force.

From Eq. 3.19–3.20 (at $\boldsymbol{F_{12}} = -\boldsymbol{F_{21}}$) we obtain

$$\boldsymbol{m_1 \upsilon_1 + m_2\, \upsilon_2 = m_1 u_1 + m_2\, u_2} \qquad (3.21)$$

$$\text{or } \boldsymbol{p_1 + p_2 = \mathbf{constant}} \qquad (3.21a)$$

From Eq. 3.21 it follows that the vector sum of the linear momentum of two bodies before interaction is equal to a vector sum of the linear momentum of these bodies after interaction. This is the law of conservation of linear momentum for two bodies which compose an isolated system.

The law of conservation of momentum states that ***if the resultant external force acting on a system of bodies is zero (the system is isolated), the total linear momentum of the system is conserved (remains constant).***

The total linear momentum of the system is a vector sum of the linear momentum of each body in the system.

***Example 3.3*. Momentum. Impulse equation.**

A golf ball of mass 0.05 kg is hit off the tee at a speed of 50 m/s. The golf club was in contact with the ball for 2 ms. Find (a) the impulse (momentum) imparted to the golf ball, and (b) the average force exerted on the ball by the golf club.

***Solution*. a)** The impulse imparted to the golf ball is given by the Eq. 3.17:

$$P = m \cdot \upsilon = 0.05 \text{ kg} \cdot 50 \text{ m/s} = 2.5 \text{ kg}\cdot\text{m/s}.$$

b) We use the impulse equation (Eq. 3.18). The initial momentum of the ball is equal to zero. Then $F \cdot t = p$.

The average force exerted on the ball by the golf club is

$$F = \frac{p}{t} = \frac{2.5 \text{ kg} \cdot \text{m/s}}{2 \cdot 10^{-3} \text{ s}} = 1.25 \text{ kN}.$$

***Example 3.4*. Conservation of momentum.**

A railroad car with a mass of 7000 kg traveling at speed of 25 m/s strikes a second railroad car with a mass of 3000 kg. If the cars lock together as a result of the collision, what is their common speed immediately after the collision?

Solution. The masses of the railroad cars are m_1 = 7000 kg and m_2 = 3000 kg, respectively; initial speed of the first railroad car υ_A = 25 m/s and of the second car (υ_B) is equal to zero. Then the initial total momentum is

$$m_1 \cdot \upsilon_A + m_2 \cdot \upsilon_B = m_1 \cdot \upsilon_A.$$

Choose the positive direction to the right side as shown in fig. 3.17. After the collision, the two cars become attached, so they will have the same speed (υ'). Then the total momentum after the collision is

$$m_1 \cdot \upsilon' + m_2 \cdot \upsilon' = (m_1 + m_2) \cdot \upsilon'.$$

According to the law of momentum conservation $m_1 \cdot \upsilon_A = (m_1 + m_2) \cdot \upsilon'$.

Then

$$\upsilon' = \frac{m_1 \cdot \upsilon_A}{m_1 + m_2} = \frac{(7000 \text{ kg}) \cdot (25 \text{ m/s})}{7000 \text{ kg} + 3000 \text{ kg}} = 17.5 \text{ m/s}.$$

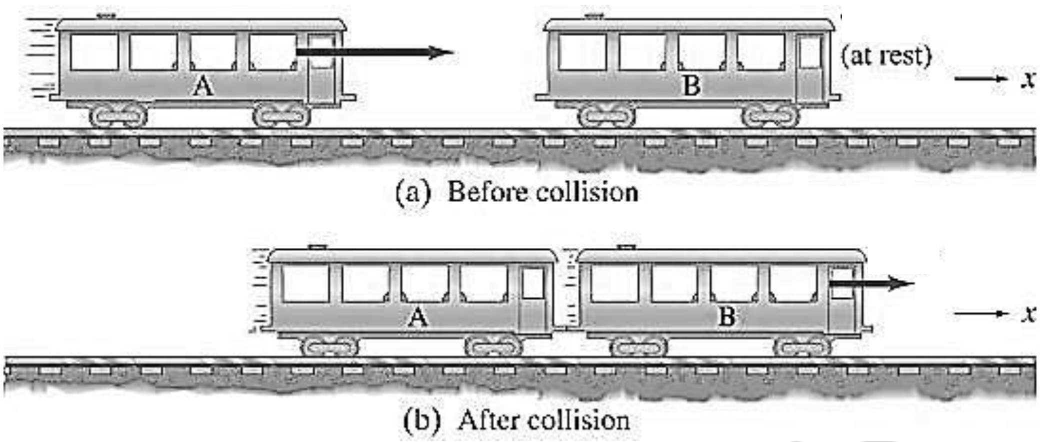


Fig. 3.17. Example 3.4. Collision of railroad cars

NOTE. *In case of equal masses (m_1 = m_2) the mutual speed after collision is half the initial speed of the first car*.

Example 3.5.

A bullet of mass 10 gram is horizontally fired from a gun of mass 5 kg with a speed of 100 m/s. What is the recoil (backward) speed of the gun?

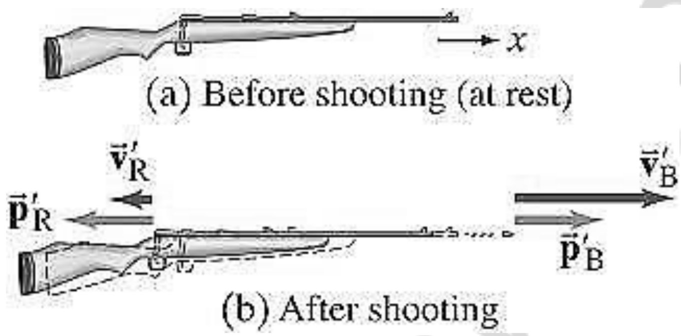


Fig. 3.18. Example 3.5. Recoil of a gun

Solution. We have the mass of bullet (m =10 g = 0.01 kg) and the mass of the gun (M = 5 kg); initial velocities of the bullet (υ_B) and gun (υ_R) are equal to zero. The final speed of the bullet υ_B' = 100 m/s. The positive direction of bullet is taken from left to right (fig. 3.18). Let υ_R' be the recoil speed of the gun. Total momentum of the gun and the bullet before the fire (the gun is at rest) is $m \cdot \upsilon_B + M \cdot \upsilon_R = 0$.

Total momentum of the gun and bullet after it is fired $m \cdot \upsilon_B' - M \cdot \upsilon_R'$.

Negative sign indicates that the direction in which the gun would recoil is opposite to that of bullet (fig. 3.18). According to the law of conservation of momentum $m \cdot \upsilon_B' - M \cdot \upsilon_R' = 0$.

Then $\upsilon_R' = \dfrac{m \cdot \upsilon_B'}{M} = \dfrac{0.01\ \text{kg} \cdot 100\ \text{m/s}}{5\ \text{kg}} = 0.2$ m/s.

The recoil speed of the gun is 0.2 m/s.

PROBLEMS

1. A 75 kg man stands in a lift. What force does the floor exerts on him when the elevator starts moving upward with an acceleration of 2.0 m/s^2 (assume g = 10.0 m/s^2). (Answer: 900 N)

2. Tendons are strong elastic fibers that attach muscles to bones. To a reasonable approximation, they obey Hooke's law. In laboratory tests on a particular tendon, it was found that, when a 250-g object was hung from it, the tendon stretched 1.25 cm. What is the stiffness constant of this tendon? (Answer: 200 N/m)

3. A brick weighing 1 kg is sliding on ice with 2 m/s. It is stopped by friction in 10 s. Calculate the constant force of friction. (Answer: 0.2 N)

4. Force of 10 N acts on a body for 40 s. Calculate the change in the momentum of the body. (Answer: 400 kg·m/s)

5. You throw a ball with a mass of 0.30 kg against a brick wall. It hits the wall and rebounds horizontally with the speed of 20 m/s. (a) If the ball is in contact with the wall for 0.010 s, find the impulse of the net force on the ball during its collision with the wall. (b) find the average horizontal force that the wall exerts on the ball during the impact. (Answer: 6 N·s, 600 N)

6. While launching a rocket of mass $2 \cdot 10^4$ kg a force of $5 \cdot 10^5$ N is applied for 20 s. What is the velocity attained by the rocket at the end of this interval. (Answer: 500 m/s)

TESTS

1. Inertia of a body directly depends upon:

a) mass b) area c) volume d) velocity

2. When a body is at rest

a) no force acts on it;
b) the force acting has no contact with it;
c) the forces acting on it balance each other;
d) none of the above.

3. A force of 2 N acting on a certain mass for 6 sec. gives it a velocity of 6 m/s. The mass is equal to:

a) 0.5 kg b) 1 kg c) 2 kg d) 4 kg

4. A force acting on a body of 10 kg produces in it an acceleration of 2 m/s^2. The force is:

a) 5 N b) 20 N c) 10 N d) none of the above

5. A man of mass m is standing on a lift which is moving downwards with acceleration a. The weight of the man is:

a) mg b) m·(g + a) c) m·(g – a) d) zero

6. In the above problem if the downward acceleration of the lift is equal to the acceleration due to gravity, then the weight of the man is:

a) mg b) m·(g + a) c) m·(g – a) d) zero

7. A spring being compressed by 0.1 m develops a restoring force 10 N. The stiffness constant of the spring is:

a) 100 N/m b) 10 N/m c) 1 N/m d) 1000 N/m

8. The momentum of the system is conserved:

a) always;
b) never;
c) only in the absence of an external force;
d) only when an external force acts.

9. A body of 2 kg is at rest. The impulse required to impart it a velocity of 8 m/s is:

a) 16 N·s b) 40 N·s c) 80 N·s d) none of the above

10. A force of 10 N acts on a body for 5 s. The change in its momentum is:

a) 2 kg·m/s b) 0.5 N·s c) 50 kg·m/s d) 500 kg·m/s

11. A force of 1 N acts on a body of mass 1 kg. The body acquires an acceleration of:

a) 1 m/s^2 b) 9.8 m/s^2 c) 1/9.8 m/s^2 d) $(9.8)^2$ m/s^2

12. A force of 6 N acts on a body at rest of mass 0.1 kg which acquires a velocity 30 m/s. The time for which the force acts is:

a) 18 s b) 5 s c) 0.5 s d) 0.3 s

13. A bomb of mass 9 kg explodes into two pieces of mass 3 kg and 6 kg. The velocity of 3 kg is 16 m/s. The velocity of 6 kg is:

a) 4 m/s b) 8 m/s c) 16 m/s d) 32 m/s

14. A body is moving with uniform momentum of 10 kg·m/s. The force acting on it is:

a) zero b) 10 N c) 0.1 N d) 100 N

15. A body of mass 2 kg moves with an acceleration of 3 m/s^2. The change in momentum in one second is:

a) 0.67 kg·m/s b) 1.5 kg·m/s c) 6 kg·m/s d) 3 kg·m/s

4. WORK. POWER. ENERGY

4.1. WORK

Work is done on an object by a force when the object moves through some distance. The **work** ***W*** done by a constant force ***F*** (constant in both magnitude and direction) is defined as a product of the magnitude of the displacement (***r = d***), the force ***F*** and cosine of the angle θ between the directions of the force and the displacement (fig. 4.1):

$$\boldsymbol{W = F \cdot d \cdot \cos\theta.} \qquad (4.1)$$

In SI units work is measured in joule (J): 1 J = 1 N·m.

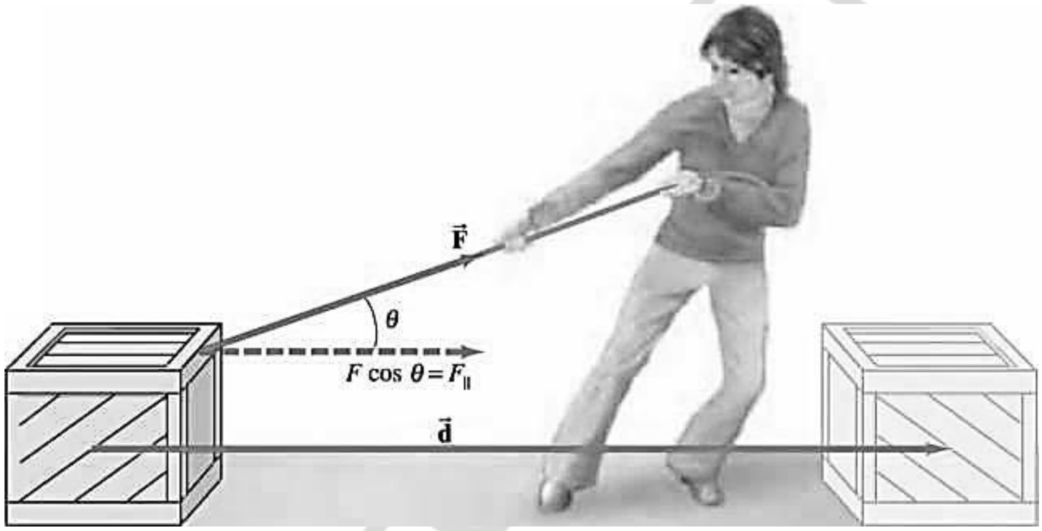


Fig. 4.1. The work done by the force ***F*** acting at an angle θ to the displacement vector ***d***

Work is *a scalar quantity*; it has only magnitude, which can be positive or negative.

For the case when the motion and the force are in the same direction (**θ = 0°** and **cos θ = 1**) ***W = F·d***. When a particular force is perpendicular to the displacement (**θ = 90°**, **cos θ = 0**), no work is done by that force.

In case of linear motion along a straight-line path in the same direction, *the magnitude of displacement is the distance* ***(d = s)***. Therefore

$$\boldsymbol{W = F \cdot s \cdot \cos\theta.} \qquad (4.2)$$

***Example 4.1*. Work.**

A person pulls a 20 kg box 10 m along a horizontal floor by a constant force F = 100 N acting at 30° angle (fig. 4.1). The floor exerts the opposite force of friction equal to 60 N. How much work does each of the following forces do on the box: a) 100-N pull, b) the friction force, c) the normal force from the floor, and gravity? d) What is the net work done on the box?

***Solution*.** A free-body diagram is similar to the Example 3.2 (fig. 3.15).

a) Setting in Eq. 4.1 F_1 = 100 N, d = 10 m and θ = 30° gives the work done by the force F_1

$$W_1 = F_1 \cdot d \cdot \cos\theta = (100\ \text{N})\cdot(10\ \text{m})\cdot\cos 30° = 866\ \text{J}.$$

b) Setting in Eq. 4.1 F = 60 N, d = 10 m and θ = 180° gives the work done by the friction force

$$W_2 = F_2 \cdot d \cdot \cos\theta = (60\ \text{N})\cdot(10\ \text{m})\cdot\cos 180° = –600\ \text{J}.$$

c) Directions of the normal force (N) and the force due to gravity (F_g) are perpendicular to the displacement (θ = 90°). Then work done by the given forces is equal to zero ($W_3 = 0$).

d) When several forces act on a body there are two ways to find the net work. One way is to use Eq. 4.1 to compute the work done by each separate force.

Then, because work is a scalar quantity, the total work done on the body by all the forces is the algebraic sum of the quantities of work done by the individual forces. An alternative way to find the total work is to compute the net force (i. e. vector sum of the forces) and then use Eq. 4.1. Let us illustrate both of these approaches.

1) The algebraic sum of the quantities of work done by the person and the friction force is

$$W_{net} = W_1 + W_2 = 866\ \text{J} – 600\ \text{J} = 266\ \text{J}.$$

2) Just the projection of the net force parallel to the displacement vector does the work on the box. It is given by

$$F_{net} = F_1 \cdot \cos\theta – F_2 = (100\ \text{N})\cdot(\cos 30°) – 60\ \text{N} = 86.6\ \text{N} – 60\ \text{N} = 26.6\ \text{N}.$$

The work done by the net force is

$$W_{net} = F_{net} \cdot \text{d} \cdot \cos\theta = (26.6\ \text{N})\cdot(10\ \text{m})\cdot\cos 0° = 266\ \text{J}.$$

4.2. POWER

Power is defined as the rate at which work is done. The average **power** $\boldsymbol{P}$ is equal to the work $\boldsymbol{W}$ done divided by the time $\boldsymbol{t}$ it takes to do it:

$$\boldsymbol{P} = \frac{\boldsymbol{W}}{\boldsymbol{t}}. \tag{4.3}$$

In SI system units, power is measured in joules per second, and this unit is given a special name, the watt (W): 1W = 1 J/s.

The work done at power of one thousand watts for one hour is equal to one kilowatt-hour (1 kWh):

$$1\ \text{kWh} = (10^3\ \text{W})\cdot(3600\ \text{s}) = 3.6 \cdot 10^6\ \text{J} = 3.6\ \text{MJ}.$$

Efficiency. An important characteristic of all engines is their overall efficiency η, defined as the ratio of the useful power output of the engine $\boldsymbol{P_{out}}$ to the power input $\boldsymbol{P_{in}}$:

$$\boldsymbol{\eta} = \frac{\boldsymbol{P_{out}}}{\boldsymbol{P_{in}}}. \tag{4.4}$$

The efficiency is always less than 1.0 because no engine can create energy, and in fact, cannot even transform energy from one form to another without some going to nonuseful forms of energy (friction, thermal energy, etc.). For example, an automobile engine converts chemical energy released in the burning of gasoline into mechanical energy that moves the pistons and then the wheels. But nearly 85 % of the input energy is "wasted" as thermal energy that goes into the cooling system or out the exhaust pipe, plus friction in the moving parts. Thus car engines are roughly only about 15 % efficient.

4.3. ENERGY

Energy can be defined as the ability to do work. Energy is measured in the same units as work: joules in SI units.

An object in motion has the ability to do work and thus can be said to have energy. The energy of motion is called *kinetic energy* (from the Greek word kinetikos, meaning "motion").

To obtain a quantitative definition for kinetic energy, let us consider a simple rigid object of mass $\boldsymbol{m}$ (treated as a particle) that is moving in a straight line with an initial speed $\boldsymbol{\upsilon_1}$. To accelerate it uniformly to a speed $\boldsymbol{\upsilon_2}$, a constant net force $\boldsymbol{F}$ is exerted on it parallel to its motion over a displacement $\boldsymbol{d}$. According to Newton's second law:

$$a = \frac{F}{m}. \tag{4.5}$$

The distance $\boldsymbol{s}$ travelled by the object during the time interval $\boldsymbol{t}$ is defined as:

$$S = \frac{\upsilon_2^2 - \upsilon_1^2}{2a}, \tag{4.6}$$

where $\boldsymbol{\upsilon_1}$ and $\boldsymbol{\upsilon_2}$ are the initial and final speed of the object, respectively.

Then the net work done on the object is

$$W = F \cdot s = m \cdot a \frac{\upsilon_2^2 - \upsilon_1^2}{2a} = \frac{m\upsilon_2^2}{2} - \frac{m\upsilon_1^2}{2}. \tag{4.7}$$

4.3.1. KINETIC ENERGY

The quantity $E_k = \dfrac{m\upsilon^2}{2}$ is defined as the kinetic energy of the object.

NOTE. *Equation (4.7) derived here for one dimensional motion with a constant force, is valid in general for translational motion of an object in three dimensions and even if the force varies*.

We can rewrite Eq. 4.7 as:

$$W = E_{k2} - E_{k1}. \tag{4.8}$$

Equation 4.8 is a useful result known as the work-energy principle. It states that the net work done on an object by the net resultant force is equal to the change in kinetic energy of the object.

If the initial speed of an object $\boldsymbol{\upsilon_1 = 0}$ and the final speed $\boldsymbol{\upsilon_2 = \upsilon}$, then

$$\boldsymbol{W = E_{k2} - E_{k1} = \frac{m\upsilon^2}{2} - 0 = \frac{m\upsilon^2}{2}}. \qquad (4.9)$$

From Eq. 4.9 it follows that in order to make the body move with some speed the work should be done upon the body which is equal to its kinetic energy.

NOTE. *The work-energy principle is a very useful reformulation of Newton's laws. It tells us that if (positive) net work* ***W*** *is done on an object, the object's kinetic energy increases by an amount* ***W****. The principle also holds true for the reverse situation: if the net work* ***W*** *done on an object is negative, the object's kinetic energy decreases by an amount* ***W****. In other words, a net force exerted on an object opposite to the object's direction of motion decreases its speed and its kinetic energy*.

The kinetic energy of a group of objects is the sum of the kinetic energies of the individual objects.

4.3.2. Potential energy

Potential energy associated with forces that depend on the position or configuration of objects relative to the surroundings. Various types of potential energy can be defined.

The common example of potential energy is ***gravitational potential energy***. A heavy brick held above the ground has potential energy because of its position relative to the Earth. The raised brick has the ability to do work, because if it is released, it will fall to the ground due to the gravitational force, and can do work.

Let us find the gravitational potential energy of an object near the surface of the Earth.

Consider the case of falling an object of mass $\boldsymbol{m}$ downwards from height $\boldsymbol{h_1}$ to $\boldsymbol{h_2}$ (fig. 4.2). The gravitational force on an object of mass $\boldsymbol{m}$ near the Earth's surface is $\boldsymbol{F_g = mg}$, where $\boldsymbol{g}$ is acceleration due to gravity. The work done by this gravitational force on an object that falls a vertical distance $\boldsymbol{h = h_1 - h_2}$ is

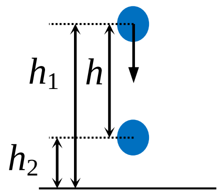


Fig. 4.2. The work done by the gravitational force

$$\boldsymbol{W_G = F \cdot s = mg \cdot (h_1 - h_2) = -(mgh_2 - mgh_1) = mgh}. \qquad (4.10)$$

Thus falling an object of mass $\boldsymbol{m}$ from height $\boldsymbol{h}$ requires an amount of work equal to $\boldsymbol{mgh}$. And once at height $\boldsymbol{h}$, the object has the ability to do an amount of work equal to $\boldsymbol{mgh}$. We can say that the work done in falling the object has been stored as ***gravitational potential energy*** $\boldsymbol{U_G = mgh}$.

We can rewrite Eq. 4.10 as:

$$\boldsymbol{W_G = -(U_{G2} - U_{G1}).} \qquad (4.11)$$

Equation 4.11 defines the change in gravitational potential energy when an object of mass $\boldsymbol{m}$ moves between two points near the surface of the Earth.

Potential energy of an object at the Earth surface is commonly set to zero ($\boldsymbol{U_{G2} = 0}$). Then the gravitational potential energy of an object, $\boldsymbol{U_G}$, at any point above the surface of the Earth at height $\boldsymbol{h}$ can be defined as

$$\boldsymbol{U_G = mgh.} \qquad (4.12)$$

Note that the gravitational potential energy is associated with the force of gravity between the Earth and the mass $\boldsymbol{m}$.

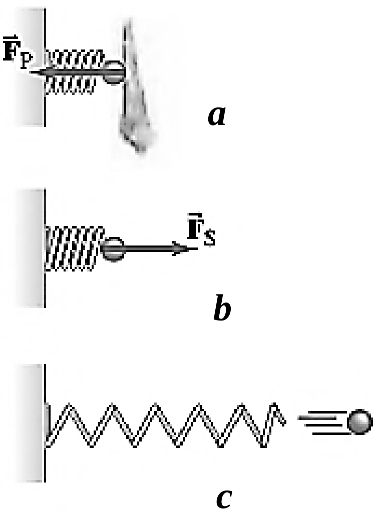


Fig. 4.3. The work done by the spring force

We consider now ***elastic potential energy*** associated with elastic materials, which includes a great variety of practical applications. Consider a simple coil spring as shown in fig. 4.3, whose mass is so small that we can ignore it. When the spring is compressed and then released, it can do work on a ball (mass $\boldsymbol{m}$). Thus the spring-ball system has potential energy when compressed (or stretched). Like other elastic materials, a spring is described by Hooke's law as long as the displacement $\boldsymbol{x}$ is not too great. To hold the spring compressed (or stretched) a distance $\boldsymbol{x}$ from its initial (unstretched) length requires the person's hand to exert a force $\boldsymbol{F_P = kx}$ on the spring (fig. 4.3, *a*), where $\boldsymbol{\kappa}$ is the spring stiffness constant. The spring pushes back with a force $\boldsymbol{F_s = -kx}$ (fig. 4.3, *b*). A spring force is thus a variable force: it varies with the displacement of the spring's free end. It can be shown that the spring-ball system has potential energy $\boldsymbol{U_{el}}$ when compressed (or stretched) an amount $\boldsymbol{x}$ from equilibrium:

$$\boldsymbol{U_{el} = \frac{kx^2}{2},} \qquad (4.13)$$

where $\boldsymbol{k}$ is the spring stiffness, $\boldsymbol{x}$ is an absolute deformation.

The quantity $\boldsymbol{E}$ equal to the sum of the kinetic energy and the potential energy of the system at any instant of time is called the total mechanical energy:

$$\boldsymbol{E = E_k + U.} \qquad (4.14)$$

If a system is isolated from its environment, there can be no energy transfers to or from it. For that case, the law of conservation of energy states: the total mechanical energy of the isolated (closed) system is conserved.

$$\boldsymbol{E = E_k + U = \text{const.}} \qquad (4.15)$$

This is called ***the principle of conservation of mechanical energy***. If the kinetic energy $\boldsymbol{E_k}$ increases, then the potential energy $\boldsymbol{U}$ must decrease by an equivalent amount to compensate. Thus the total energy $\boldsymbol{E_k + U}$ remains constant.

This is one of the most important principles in physics. It is called *the law of conservation of energy* and can be stated as follows: ***The total energy is neither increased nor decreased in any process***. *Energy can be transformed from one form to another, and transferred from one object to another, but the total amount remains constant.*

When the external forces do work on a body, the change in the total mechanical energy of the system is equal to the work done by external forces ($\boldsymbol{W}_{\mathbf{ext}}$):

$$\boldsymbol{E}_2 - \boldsymbol{E}_1 = \boldsymbol{W}_{\mathbf{ext}}. \qquad (4.16)$$

In case when nonconservative forces (such as a friction force) act within the closed system we obtain:

$$\boldsymbol{E}_2 - \boldsymbol{E}_1 = \boldsymbol{W}_{\mathbf{fric}}. \qquad (4.17)$$

Example 4.2. **Conservation of energy.**

A rock with a mass of 2 kg is falling down due to the Earth's gravity from a height h = 2.5 m above the ground (fig. 4.4). What are (a) the rock's speed when it has fallen to 1.5 m above the ground; (b) the rock's speed just before it hits the ground? (c) the potential energy of the rock at the moment of release and the kinetic energy of the rock just before it hits the ground.

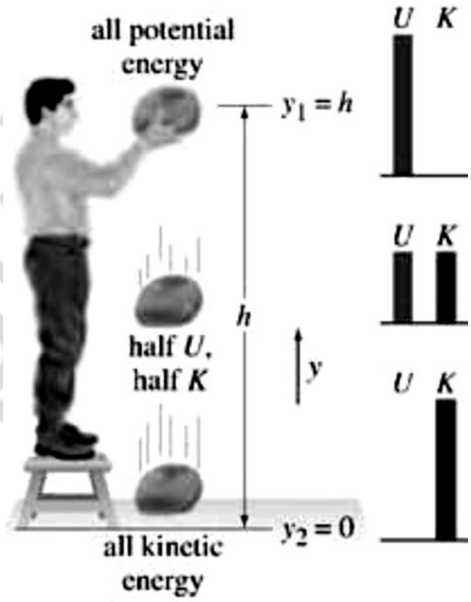


Fig. 4.4. Example 4.2. Note bar graph represents the change in potential U and kinetic E_k energy

Solution. We apply Eqs. 4.9–4.10 and the law of conservation of mechanical energy (Eq. 4.15) with only gravitational force acting on the rock. We choose the ground as the reference level. According to the Eq. 4.15 the total mechanical energy of the rock at any point along the path is constant. Then

$$\frac{m\upsilon_1^2}{2} + mgh_1 = \frac{m\upsilon_2^2}{2} + mgh_2.$$

where υ_1 is the rock's speed at the position h_1 above the ground and υ_2 is its speed at some other point h_2.

a) At the initial moment (point y_1 in fig. 4.4) position is h_1 = 2.5 m and υ_1 = 0. The rock's initial kinetic energy in equal to zero and total mechanical energy is equal to potential energy. Hence $mgh_1 = \frac{m\upsilon_2^2}{2} + mgh_2$.

We need to find the rock's speed at the height h_2. Setting $\upsilon_1 = 0$, h_2 = 1.5 m and solving for υ_2 we find

$$\upsilon_2 = \sqrt{2g(h_1 - h_2)} = \sqrt{2 \cdot 9.8 \text{ m/s}^2 (2.5 \text{ m} - 1.5 \text{ m})} = 4.3 \text{ m/s}.$$

The rock's speed 1.5 m above the ground is 4.3 m/s.

b) Just before rock hits the ground (point y_2) the height is equal to zero (h_2 = 0) and total mechanical energy is equal to the kinetic energy.

Setting h_2 = 0 in the above equation gives

$$\upsilon_2 = \sqrt{2gh_1} = \sqrt{2(9.8 \text{ m/s}^2)(2.5 \text{ m})} = 7 \text{ m/s}.$$

The rock's speed just before it hits the ground is 7 m/s.

NOTE. *The speed of the rock is independent of the rock's mass*.

c) The potential energy U_G of the rock is given by Eq. 4.10. It is equal to the initial mechanical energy of the rock. Setting h = 2.5 m in Eq. 4.10 we calculate:

$$W_G = mgh = (2 \text{ kg})\cdot(9.8 \text{ m/s}^2)\cdot(2.5 \text{ m}) = 49 \text{ J}.$$

Just before the rock hits the ground total mechanical energy is equal to the kinetic energy $E = E_k$. According to the law of conservation of mechanical energy it is equal to the initial mechanical energy of the rock, i. e. $E = E_k = W_G$ = 49 J.

NOTE. *As the rock falls, the potential energy decreases, but the rock's kinetic energy increases to compensate, so that the sum of the two remains constant. Just before the rock hits the ground all of the initial potential energy will have been transformed into kinetic energy* (fig. 4.4).

PROBLEMS

1. A tow truck pulls a car 5 km along a horizontal roadway using a cable having a tension of 900 N. How much work does the cable do on the car if it pulls horizontally? (Answer: 4500 kJ)

2. A body is under the action of a force 5 N moves through 10 m in a straight line. If work done is 25 J what is the angle at which force acts with the direction of motion. (Answer: 60°)

3. A factory worker pushes a 50-kg crate a distance of 5 m along a level floor at constant velocity by pushing horizontally on it. The coefficient of kinetic friction between the crate and the floor is 0.25. (a) What magnitude of force must the worker apply? (b) How much work is done on the crate by this force? (c) How much work is done on the crate by friction? (e) What is the total work done on the crate? (Answer: 122.5 N, 612.5 J, –612.5J, 0 J)

4. The human heart is a powerful and extremely reliable pump. Each day it takes in and discharges about 7500 L of blood. Assume that the work done by the heart is equal to the work required to lift this amount of blood a height equal to that of the average woman (1.64 m). The density of blood is ρ = 1060 kg/m^3. (a) How much work does the heart do in a day? (b) What is the heart's power output in watts? (Answer: 127.772 kJ, 1.5 W).

5. How long will it take a 1250 Watts motor to lift a 400-kg piano to a sixth-story window 15 m above? (Answer: 47 s)

6. The maximum height a typical human can jump from a crouched start is about 50 cm. By how much does the gravitational potential energy increase for a 80-kg person in such a jump? (Answer: 392 J)

7. A 150-g baseball is dropped from a tree 15 m above the ground. (a) With what speed would it hit the ground if air resistance could be ignored? (b) If it actually hits the ground with the 10 m/s, what is the average force of air resistance exerted on it? (Answer: 17.2 m/s, 0.97 N)

8. A spring has a stiffness constant k of 80 N/m. How much must this spring be compressed to store 40 J of potential energy? (Answer: 1 m)

TESTS

1. Work is:

a) scalar quantity   b) vector quantity
c) both scalar and vector   d) none of the above

2. Power of the body is given:

a) total capacity of doing work   b) rate of doing work
c) product of work and time   d) none of the above

3. Energy is defined as:

a) total capacity of doing work   b) rate of doing work
c) product of work and time   d) none of the above

4. An object is thrown vertically upwards. As it rises its total energy:

a) decreases   b) increases
c) remains constant   d) sometimes decreases, sometimes increases

5. A force 100 *N* is required to move a body with a velocity of 10 m/s. The power developed is:

a) 50 watts   b) 1000 watts   c) 10 watts   d) 100 watts

6. A mass of 100 kg rests on a smooth horizontal surface. Energy needed to accelerate it from rest to a velocity of 10 m/s is:

a) 5000 J   b) 500 J   c) 50000 J   d) 50 J

7. Potential energy cannot be expressed in:

a) J   b) N·m   c) N·s   d) W·s

8. Two bodies of masses m_1 and m_2 have the same momenta. The ratio of their kinetic energies is:

a) $m_1:m_2$ b) $m_2:m_1$ c) $\overline{m_1} : \overline{m_2}$ d) $m_1^2:m_2^2$

9. When velocity of body is doubled, its kinetic energy:

a) doubled   b) remains the same
c) becomes 4 times   d) none of the above

10. Work can be:
a) positive only
b) negative only
c) both positive and negative
d) neither positive nor negative

11. A ship of mass $5 \cdot 10^7$ kg is acted upon a force of $20 \cdot 10^4$ *N* by an engine which moves it through 5 m. If resistance of water is negligible, the speed of the ship is:
a) 10 m/s b) 1 m/s c) 0.2 m/s d) 5 m/s

12. When mass and velocity of a body is doubled its kinetic energy is
a) doubled b) four times c) eight times d) sixteen times

13. Work done against friction is:
a) negative b) positive c) zero d) none of the above

14. A body moves a distance of 20 m along straight path when a force of $5/\sqrt{2}$ *N* acts on it. If work done is 50 *J*, at what angle with the direction of motion the force acts?
a) 90° b) 60° c) 0° d) 45°

5. MECHANICAL OSCILLATIONS AND WAVES

5.1. MECHANICAL OSCILLATIONS

Movements or changes of the state of some system, which periodically (at regular intervals) repeat are called *oscillations (or vibrations).* If an object vibrates or oscillates back and forth over the same path, each cycle taking the same amount of time, the motion is called periodic.

5.1.1. CHARACTERISTICS OF OSCILLATIONS

Characteristics of oscillations are as follows displacement, amplitude, period and frequency.

The distance ***x*** of the object from the equilibrium point at any instant of time is called ***the displacement***. It is measured in metres ***m***.

The maximum displacement ***A*** (the greatest distance from the equilibrium point) is called ***the amplitude***. The units of amplitude are the same as for displacement (***m***).

One cycle is the complete motion from some initial point back to the same point. The ***period T*** is defined as the time required to complete one cycle. It is measured in seconds (***s***).

The ***frequency*** (***f*** or ν) is the number of complete cycles per second. Frequency and period are inversely related:

$$f = \frac{1}{T}. \qquad (5.1)$$

In the SI system frequency is measured in hertz (Hz), where 1 Hz = 1 cycle per second (s^{-1}).

5.1.2. SIMPLE HARMONIC MOTION

One of the most important types of periodic motion is ***simple harmonic motion*** (SHM). In such motion the displacement ***x*** of a particle from its equilibrium position is described by a sine (or cosine) function of time ***t****:*

$$\boldsymbol{x(t) = A \sin \varphi = A \sin(\omega t + \varphi_0)} \qquad (5.2)$$

or

$$\boldsymbol{x(t) = A \cos \varphi = A \cos(\omega t + \varphi_0)},$$

where ***A*** is the amplitude, $\boldsymbol{\varphi = \omega t + \varphi_0}$ is the phase, $\boldsymbol{\varphi_0}$ is the initial phase, **ω** is the angular frequency.

The phase of oscillations characterizes the state of the vibrational system at any instant of time.

The above equations are called ***the equations of simple harmonic oscillations***.

The angular frequency **ω** is related to the period ***T*** and the frequency of the motion ***f*** by the equation:

$$\boldsymbol{\omega = 2\pi f = \frac{2\pi}{T}}. \qquad (5.3)$$

The SI unit of angular frequency is the radian per second (rad/s).

NOTE. *SHM is the motion of a body under the influence of elastic or a similar force that is proportional to the body's displacement but has the opposite direction.*

Simple harmonic motion can be represented graphically as shown in fig. 5.1.

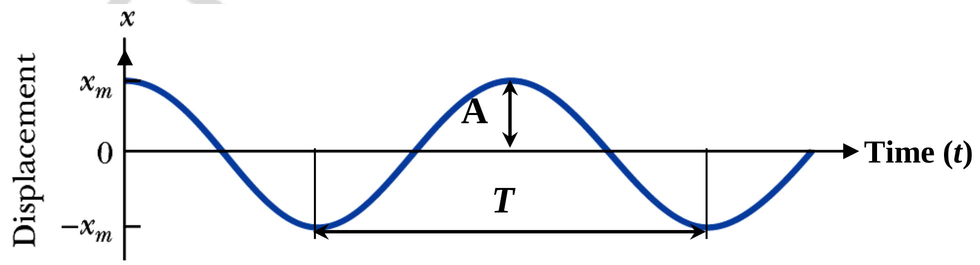


Fig. 5.1. Simple harmonic vibration

5.1.3. EXAMPLES OF MECHANICAL OSCILLATIONS

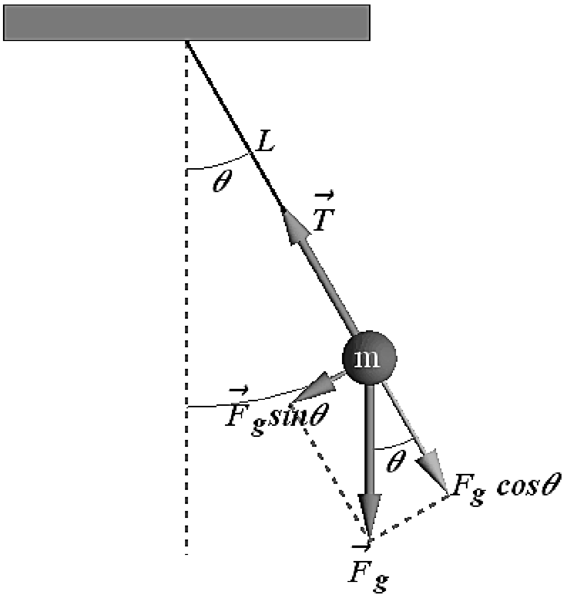


Fig. 5.2. A simple pendulum

The simple pendulum oscillations. The simple pendulum consists of a particle of mass ***m*** (called the bob of the pendulum) suspended from one end of an unstretchable, massless string of length ***L*** that is fixed at the other end (fig. 5.2). The bob is free to swing back and forth in the plane of the page, to the left and right of a vertical line through the pendulum's pivot point. The forces acting on the bob are the force from the string ***T*** and the gravitational force $\boldsymbol{F_g}$, where the string makes an angle θ with the vertical line (fig. 5.2). If the bob swings through only small angles, its motion is approximately simple harmonic motion. In other words, the restriction is that the angular amplitude of the motion (the maximum angle of swing) must be small (θ < 5°).

The period ***T***, the frequency ***f*** and the angular frequency ω of the simple pendulum are defined as

$$\boldsymbol{T} = 2\pi\sqrt{\frac{\boldsymbol{L}}{\boldsymbol{g}}} \quad \text{or} \quad \boldsymbol{f} = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{g}}{\boldsymbol{L}}}, \quad \omega = \sqrt{\frac{\boldsymbol{g}}{\boldsymbol{L}}}, \qquad (5.4)$$

where ***L*** is the length of the pendulum string, ***g*** is the acceleration due to gravity.

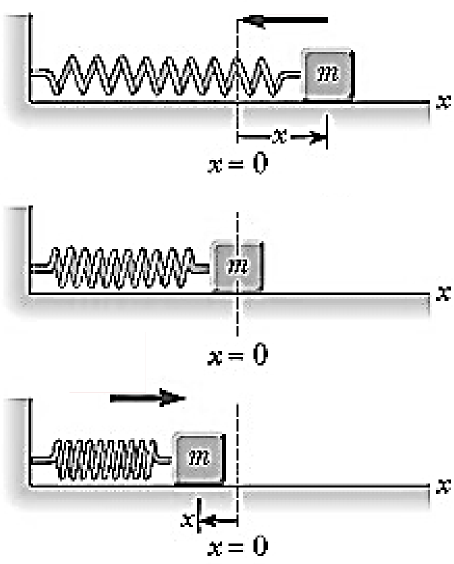


Fig. 5.3. A linear simple harmonic oscillator

Spring pendulum oscillations. Spring pendulum consists of a body of mass ***m*** attached to a horizontal (or vertical) spring as shown in fig. 5.3. Such block-spring system is called a linear simple harmonic oscillator (or linear oscillator), where linear indicates that a restoring elasticity force $\boldsymbol{F_{el}}$ is proportional to the displacement ***x*** to the first power ($\boldsymbol{F_{el}} = -\boldsymbol{kx}$). The body moves in simple harmonic motion under the action of an elastic force $\boldsymbol{F_{el}}$ once it has been either pulled or pushed away from the equilibrium position and released.

The period ***T***, the frequency ***f*** and the angular frequency ω of the spring pendulum are defined as

$$\boldsymbol{T} = 2\pi\sqrt{\frac{\boldsymbol{m}}{\boldsymbol{k}}}, \quad \boldsymbol{f} = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{k}}{\boldsymbol{m}}}, \quad \omega = \sqrt{\frac{\boldsymbol{k}}{\boldsymbol{m}}}, \qquad (5.5)$$

where ***m*** is the mass of a body, ***k*** is the spring stiffness.

***Example 5.1*. Equation of simple harmonic oscillations.**

A body with a mass of 500 g is undergoing harmonic oscillations described by the following equation $x = 10\sin(5\pi \cdot t + \frac{\pi}{6})$, where t is in seconds and x in meters. Find (a) the amplitude, (b) the frequency and the period, (c) the position of a body at the initial instant of time $t = 0$, (d) the spring stiffness.

Solution. **a)** We use Eqs. 5.1–5.3, from which it follows that the amplitude is $A = 10$ m, the angular frequency is $\omega = 5\pi$.

b) The frequency of the oscillations is $f = \frac{\omega}{2\pi} = \frac{5\pi}{2\pi} = 2.5$ Hz. Then the period is equal to $T = \frac{1}{f} = \frac{1}{2.5} = 0.4$ s.

c) The position of a body at $t = 0$ is $x(0) = 10\sin(\frac{\pi}{6}) = 5$ m.

d) The spring stiffness is $k = \omega^2 m = 246.5 \cdot 0.5 = 123.3$ N/m.

***Example 5.2*. Equation of simple harmonic oscillations.**

What is the equation describing the motion of a mass at the end of a spring which position at the initial instant of time $t = 0$ is $x_0 = 5$ cm, and whose period is $T = 0.314$ *s* and amplitude is $A = 10$ cm?

Solution. We use Eq. 5.2: $x(t) = A\sin(\omega t + \varphi_0)$.

Angular frequency is given by

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.314} = 20 \text{ rad/s.}$$

At the initial instant of time $t = 0$ the position of a body is $x_0 = 5$ cm. Amplitude is $A = 10$ cm. Then from Eq. (5.2) it follows that

$$\sin\varphi_0 = \frac{x_0}{A_0} = \frac{5}{10} = \frac{1}{2}$$

or

$$\varphi_0 = \frac{\pi}{6}.$$

The equation of simple harmonic oscillations is given by

$$x = 10\sin(20t + \frac{\pi}{6}) \text{ (cm).}$$

5.2. MECHANICAL WAVES

Mechanical wave is the process of the vibrations' propagation in the elastic medium. In other words waves are disturbances which propagate through a medium. Particles form the elastic medium and their vibrations produce the wave. Waves can be viewed as a transfer of energy and particles' vibrations through the medium but not the straight-line movement of the particles.

The mechanical wave is called ***longitudinal*** if the direction of the displacement of the medium particles is along the direction of the wave propagation. In a ***transverse*** wave the direction of the displacement of the medium particles is perpendicular to the direction of the wave's motion. Longitudinal mechanical waves can propagate in different media (except of vacuum), and transverse waves can propagate just in solid.

Waves have peaks and troughs. The height of a peak and depth of a trough is called the amplitude of the wave (fig. 5.4).

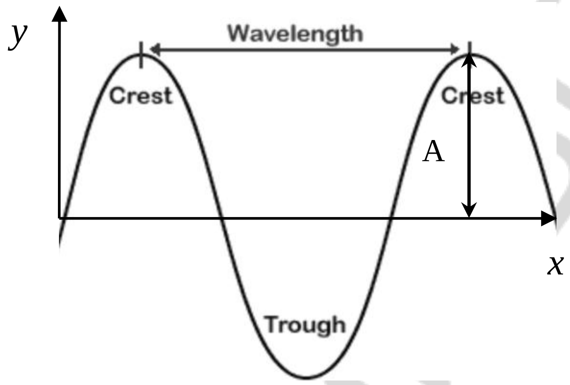


Fig. 5.4. A schematic wavelength representation

The distance between two adjacent peaks (or troughs) is the same. This distance which is a characteristic of the wave is called *the wavelength* **λ**. The units are metres ***m***. *The wavelength is the distance between any two adjacent points which are in phase.* The wavelength **λ** is the distance that the wave propagates in the medium at speed $\boldsymbol{\upsilon}$ for the time equal to the period ***T***:

$$\lambda = \upsilon T = \frac{\upsilon}{f}. \qquad (5.6)$$

Acoustic waves are mechanical longitudinal waves, which propagate in the elastic medium and have frequencies from the lowest ones to 10^{12}–10^{13} Hz.

Sound (audible) waves have frequencies about 16 Hz to 20 000 Hz. Sound having frequencies above the range of human hearing is called ultrasound. Waves having frequencies below 16 Hz are called infrasound.

Speed of acoustic waves is dependent on the properties of the medium through which they propagate (e. a. temperature, elasticity, density).

In summary a sound is a mechanical wave that moves through a medium as particles in the medium are displaced relative to each other. The speed of sound is different in different materials; in general, it is slowest in gases (υ = 340 m/s in air), faster in liquids (υ = 1450 m/s in water), and fastest in solids (υ = 4900 m/s in iron).

***Example. 5.3*. Wavelength and frequency.** (a) What is the range of the wavelengths of audible sound in air? (b) What is the frequency of ultrasound corresponding to the wavelength equal to 1 mm? Assume the speed of sound in air equal to 340 m/s.

***Solution*.** We use the Eq. 5.6 **a)** The range of audible frequencies is from about 16 Hz to 20 000 Hz. Then from Eq. 5.6 we obtain

$$\lambda_1 = \frac{\upsilon}{f_1} = \frac{340\ \text{m/s}}{16\ \text{Hz}} = 21.25\ \text{m}$$

$$\lambda_2 = \frac{\upsilon}{f_2} = \frac{340\ \text{m/s}}{20\ 000\ \text{Hz}} = 17\ \text{mm}.$$

The range of the wavelengths of audible sound in air is from 17 mm to 17 m.

b) The frequency of ultrasound in air corresponding to the wavelength equal to 1 mm is

$$f = \frac{\upsilon}{\lambda} = \frac{340\ \text{m/s}}{0.001\ \text{m}} = 0.34\ \text{MHz}.$$

PROBLEMS

1. A particle is executing simple harmonic vibrations with the period T = 12 s and the initial phase φ_0 = 0. How much time does it take to travel a distance equal to half of its amplitude? (Answer: 1 s)

2. A pendulum is first vibrated on the surface of earth. Its period is T. It is then taken to the surface of moon where acceleration due to gravity is 1/6th that on earth. What is its period? (Answer: $\sqrt{6}T$)

3. A spring is loaded with mass m and the period of oscillations is T. It is then cut into 4 equal parts. What is the period of oscillations of each part? (Answer: T/2)

4. In diagnostic ultrasound imaging the speed of sound is assumed to be 1540 m/s in soft tissues. If the ultrasound wave has the wavelength of 0.1 mm, what is its time period and frequency? (Answer: 0.065 μs, 15.4 MHz)

TESTS

1. A metal sphere is suspended to a spring. It oscillates with frequency f. It is then taken to the moon where the acceleration of gravity (g) becomes 1/6th of its value on the earth. What is the frequency of oscillations?

a) 6f b) $\sqrt{6}f$ c) the same d) f/6

2. A simple pendulum consists of a bob of a radius r and mass m and its period is 2 sec. When its bob is replaced by a bob of a mass of 2 m but the same radius r, the time period of motion is:

a) 4 sec b) 2 sec c) 1 sec d) 8 sec

3. A mass m is suspended to a spring of length L and stiffness constant k. The frequency of vibration is f_1. The spring is cut into two equal parts and each half is loaded with the same mass m. The new frequency f_2 is given by:

a) $f_2 = \sqrt{2} f_1$ b) $f_2 = \frac{f_1}{\sqrt{2}}$ c) $f_2 = \frac{f_1}{2}$ d) $f_2 = 2f_1$

4. The equation of motion for a body executing simple harmonic vibrations is given by $x = 0.5 \sin(10\pi t + 5)$. The frequency is given by:

a) 5 Hz b) 1 Hz c) 0.5 Hz d) 5π Hz

5. For a body of mass m attached to the spring the stiffness constant (k) is given by (ω is the angular frequency):

a) $\frac{m}{\omega^2}$ b) $m\omega^2$ c) $m^2 \omega$ d) $m^2\omega^2$

6. Simple harmonic vibrations is given by $x = A \sin(\omega t + \varphi_0)$. When a particle is at its positive extreme, its phase relative to equilibrium position is:

a) π b) π/2 c) zero d) 2π

7. Simple harmonic vibrations is given by $x = A \sin(\omega t + \varphi_0)$. If $\varphi_0 = \pi/6$, the body started oscillating from:

a) $x = 0$ b) $x = \frac{\sqrt{3}}{2} A$ c) $x = A/2$ d) $x = A/\sqrt{2}$

8. Sound waves can not be propagated through:

a) a gas b) a liquid c) vacuum d) a solid

9. A source of sound vibrates according to the equation $x = 0.03 \sin 100\pi t$. The speed of the wave is 1.5 km/s. What is the wavelength of the wave?

a) 60 m b) 30 m c) 15 m d) 45 m

10. The sound generator dipped in sea is sending waves of wavelength 2.5 m and frequency 580 Hz. The speed of sound in sea water is:

a) 1250 m/s b) 1450 m/s c) 1650 m/s d) 1050 m/s

11. The speed of sound is the largest in:

a) air b) water c) vacuum d) steel

12. Two persons cannot hear each other on the surface of moon because the moon has:

a) craters b) no atmosphere
c) rocks which absorb sound d) dust suspended all around

13. The range of frequencies audible to human ear is:

a) 16 Hz to 20000 Hz b) 16 Hz to 2000 Hz
c) 100 Hz to 10000 Hz d) 16 kHz to 20000 kHz

6. STATICS

Statics is concerned with the analysis of loads (i. e. forces) acting on and within structures that are in equilibrium. Equilibrium means a state of balance. A body is *in static equilibrium* when it is at rest relative to a given frame of reference.

In practice, many structural analyses in biomechanics are performed based on the assumption of static equilibrium.

In this section we will concern with rigid solid bodies which do not bend, stretch, or squash when forces act on them.

Let us consider a rigid solid body (fig. 6.1) which can rotate about an axis that is perpendicular to the plane of the figure and passes this plane through point ***O***. Two forces, $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$, act on the body in the plane of the figure. Such a body is called a lever. *By definition a lever is a rigid* solid *body (commonly a bar) free to rotate about a fixed point called the fulcrum (point* ***O****).* The efficiency of the force to cause a rotation about the fulcrum depends on the ***torque T*** *(or moment)* of the force with respect to point ***O***. The torque of the force with respect to the point ***O*** is defined as

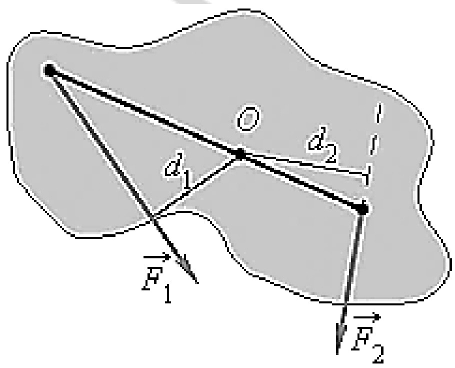


Fig. 6.1. Lever arms of the forces and

$$\boldsymbol{T = F \cdot d}, \qquad (6.1)$$

where ***F*** is the magnitude of the force, ***d*** is the *perpendicular distance* between point ***O*** and the line of action of the force ***F*** (i. e. the line along which the force vector lies) (fig. 6.1).

The distance ***d*** is called the *lever arm (or moment arm)* of force ***F*** with respect to point ***O***. The lever arms of the forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ in fig. 6.1 are the distances $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$, respectively. If the line of action of the force passes through point ***O***, so the lever arm for this force is zero and its torque with respect to ***O*** is zero. Such a force can not cause a rotation of the lever. A force of a given magnitude has the larger torque when it has the larger lever arm, this is the principle of levers.

The force $\boldsymbol{F_1}$ in fig. 6.1 tends to cause *counterclockwise* rotation about ***O***, while $\boldsymbol{F_2}$ tends to cause *clockwise* rotation. *Counterclockwise torques are commonly considered to be positive and clockwise torques are negative*, then the magnitudes of the torques of the forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ about ***O*** are

$$\boldsymbol{T_1 = +F_1 \cdot d_1 \; T_2 = -F_2 \cdot d_2}. \qquad (6.2)$$

When several torques act on a body, the net torque (or resultant torque $\boldsymbol{T}_{\text{net}}$) is the sum of the individual torques:

$$\boldsymbol{T}_{\text{net}} = \sum_{i=1}^{n} \boldsymbol{T}_{i}. \qquad (6.3)$$

The SI unit of torque is the newton-meter (N·m).

6.1. CONDITIONS FOR EQUILIBRIUM

We learned that a particle is in *equilibrium* (i. e. the particle does not accelerate) in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero. For an *extended* body, considered above, the equivalent statement is that the center of mass of the body has zero acceleration if *the vector sum of all external forces acting on the body is equal zero*. This is called the **first condition for equilibrium.**

$$\sum \overrightarrow{\boldsymbol{F}_{i}} = \boldsymbol{0} \ \text{(vector form)}$$

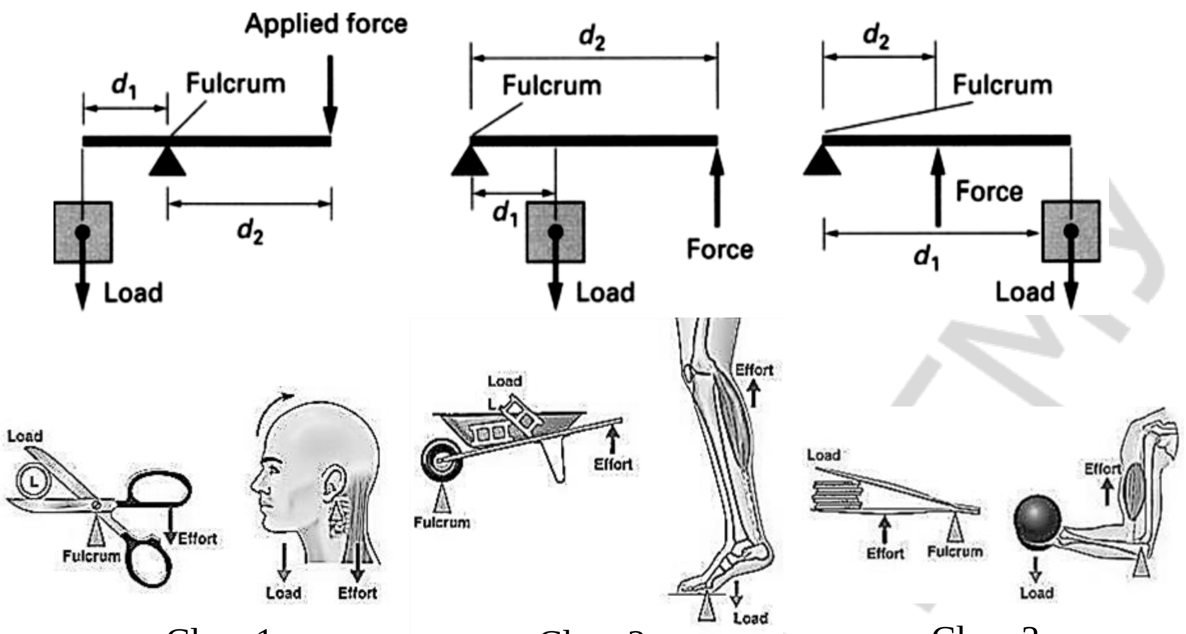
$$\text{or} \quad \boldsymbol{F}_{x} = \boldsymbol{0}, \quad \boldsymbol{F}_{y} = \boldsymbol{0}, \quad \boldsymbol{F}_{z} = \boldsymbol{0} \ \text{(vector projection form).} \qquad (6.4)$$

A second condition for an extended body to be in equilibrium is that the body must have no tendency to rotate. This condition is based on the dynamics of rotational motion in exactly the same way that the first condition is based on Newton’s first law. A rigid body in equilibrium can not have any tendency to start rotating about any point, if *the algebraic sum of the torques due to all the external forces acting on the body with respect to any specified point, is equal to zero*. This is **the second condition for equilibrium**:

$$\sum_{i=1}^{n} \boldsymbol{T}_{i} = \boldsymbol{0}. \qquad (6.5)$$

Levers. Levers are used to lift loads in an advantageous way and to transfer movement from one point to another. Levers are classified regarding the relationship between their components (i. e. fulcrum, load and applied force). There are three classes of levers, as shown in fig. 6.2.

In a Class 1 lever the fulcrum is located between the applied force and the load. A scissors and lifting a head off the chest are the examples of a Class 1 lever (fig. 6.2). In a Class 2 lever, the fulcrum is at one end of the bar; the force is applied to the other end; and the load is situated in between. A wheelbarrow and raising up onto the toes are examples of a Class 2 lever. A Class 3 lever has the fulcrum at one end and the load at the other. The force is applied between the two ends. Many of the limb movements, for example, flexion at the elbow are performed by Class 3 levers.



Class 1 Class 2 Class 3

Fig. 6.2. Three classes of levers

According to the Eq. 6.5 condition for equilibrium for all three types of levers is given by

$$\boldsymbol{T_1 - T_2 = 0} \quad \text{or} \quad \boldsymbol{F_1 \cdot d_1 - F_2 \cdot d_2 = 0.} \tag{6.6}$$

From Eq. 6.6 it follows that

$$\frac{\boldsymbol{F_1}}{\boldsymbol{F_2}} = \frac{\boldsymbol{d_2}}{\boldsymbol{d_1}} \quad \text{or} \quad \boldsymbol{F_2} = \boldsymbol{F_1}\frac{\boldsymbol{d_1}}{\boldsymbol{d_2}}, \tag{6.7}$$

where $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ are the lengths of the lever arms as shown in fig. 6.2, $\boldsymbol{F_2}$ is the applied force required to balance a load force $\boldsymbol{F_1}$. If $\boldsymbol{d_1}$ is less than $\boldsymbol{d_2}$, the applied force $\boldsymbol{F_2}$ is smaller than the load force $\boldsymbol{F_1}$.

The ratio of the load force to the applied force is called ***the mechanical advantage of the lever*** ($\boldsymbol{M} = \frac{\boldsymbol{F_1}}{\boldsymbol{F_2}} = \frac{\boldsymbol{d_2}}{\boldsymbol{d_1}}$). In dependence on the distances from the fulcrum, the mechanical advantage of a Class 1 lever can be greater or smaller than one. By placing the load close to the fulcrum, with $\boldsymbol{d_1}$ much smaller than $\boldsymbol{d_2}$, a very large mechanical advantage can be obtained with a Class 1 lever. In a Class 2 lever, $\boldsymbol{d_1}$ is always smaller than $\boldsymbol{d_2}$, therefore, the mechanical advantage of a Class 2 lever is greater than one. The situation is opposite in a Class 3 lever. Here $\boldsymbol{d_1}$ is larger than $\boldsymbol{d_2}$, therefore, the mechanical advantage is always less than one.

6.2. TYPES OF EQUILIBRIUM

There are three types of static equilibrium: stable, unstable and neutral.

Stable static equilibrium means that with small deviations of the body from the equilibrium state, forces or moments of forces emerge which tend to return the body to the state of equilibrium. A ball located at bottom of a spherical deepening is in a state of stable equilibrium (fig. 6.3).

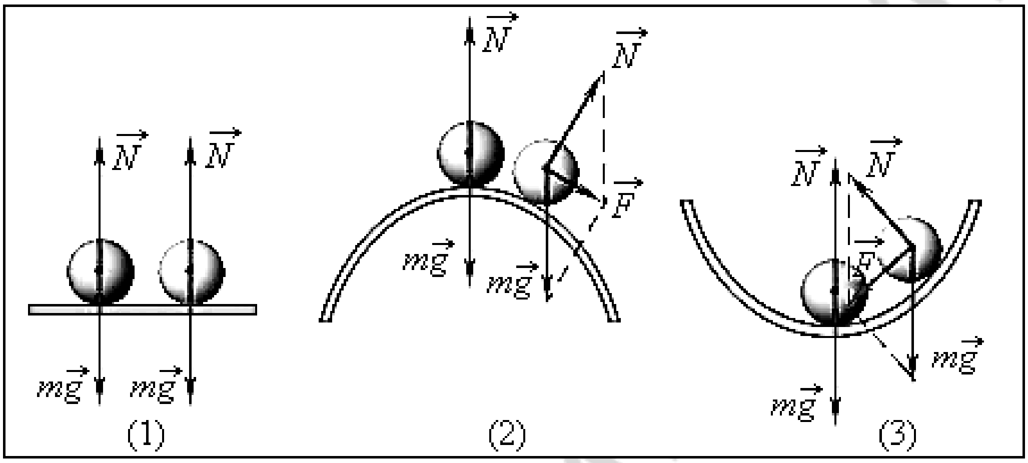


Fig. 6.3. Different types of equilibrium of the ball on a support: 1 — neutral; 2 — unstable; 3 — stable

Unstable static equilibrium means that, with a small deviation of the body from the equilibrium state, forces emerge which tend to increase this deviation. A ball located at the top of a sphere is an example of unstable equilibrium.

Neutral equilibrium means that, with a small deviation, the body remains in equilibrium. An example is a ball lying on a flat horizontal surface.

Minimum potential energy is correspondent to the state of stable equilibrium as compared to the adjacent states. Potential energy is larger in unstable equilibrium state than in the adjacent states. In case of neutral equilibrium potential energy is the same as in the adjacent states.

6.3. CENTER OF MASS AND CENTER OF GRAVITATION

When an object rotates, or several parts of a system of objects move relative to one another, there is one point that moves in the same path that a particle would move if subjected to the same net force. This point is called the *center of mass (abbreviated CM)*. The general motion of an extended object (or system of objects) can be considered as the sum of the translational motion of the CM, plus rotational, vibrational, or other types of motion about the CM.

For symmetrically shaped objects of uniform composition (such as spheres, cylinders, and rectangular solids) the CM is located at the geometric center of the object.

The center of gravity (CG) of an object is the point at which the force of gravity can be considered to act (fig. 6.4). The force of gravity actually acts on all the different parts or particles of an object, but for purposes of determining the translational motion of an object as a whole, we can assume that the entire weight of the object (which is the sum of the weights of all its parts) acts at the CG.

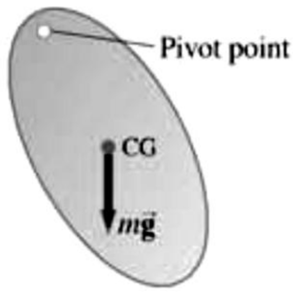


Fig. 6.4. Determination of the center of mass of a flat uniform body

Commonly the center of gravity and the center of mass are at the same point.

Center of mass or center of gravity of an extended object can be easily determined experimentally. If an object is suspended from any point, it will swing (fig. 6.4) due to the force of gravity on it, unless it is placed so its CG lies on a vertical line directly below the point from which it is suspended.

The algebraic sum of the torques due to the forces of gravity acting on all particles of the body with respect to the center of gravity is equal to zero.

The subject of statics is important because it allows us to calculate certain forces acting on (or within) a structure when some of the forces on it are already known. Let us consider several examples.

***Example 6.1*. The first class lever.**

The arms of a horizontal lever are d_1 = 25 cm and d_2 = 2 m long at opposite sides of the fulcrum (fig. 6.2). The shorter arm is loaded with the weight of 500 N at the end. (a) What force should be applied at the lever longer arm to balance the load? (b) What is the advantage of the lever?

Solution. We use Eq. 6.7. Setting, d_1 = 25 cm = 0.25 m, d_2 = 2 m, F_1 = 500 N, we calculate F_2:

$$F_2 = F_1 \frac{d_1}{d_2} = 500\text{ N}\left(\frac{0.25\text{ m}}{2\text{ m}}\right) = 62.5\text{ N}.$$

Advantage of the lever is $M = \frac{F_1}{F_2} = \frac{d_2}{d_1} = \frac{2\text{ m}}{0.25\text{ m}} = 8$ times.

The force required to balance the load is 62.5 N.

***Example 6.2*. Balance of a horizontal beam.**

A uniform beam of length L and mass m = 2 kg rests on two fulcrums as shown in fig. 6.5. A uniform block of mass M = 3.5 kg is at rest on the beam at a distance $L/4$ from its left end. Find the reactions forces exerted on the beam by the right and left fulcrums.

Solution. A free-body diagram for the system consisting of the beam and the block is shown in fig. 6.5. The forces acting on the system are as follows: the force of gravity of the beam $\vec{F}_{\text{g,beam}} = m\vec{g}$ and of the block $\vec{F}_{\text{g,block}} = M\vec{g}$

and the reaction forces F_l and F_r at the left and right ends of the beam, respectively.

We apply two conditions for equilibrium (Eqs. 6.4–6.5) and solve this problem in two equivalent ways.

From the Eq. 6.4 (balance of forces) we obtain:

$$m\vec{g} + M\vec{g} + \vec{F_l} + \vec{F_r} = 0.$$

Let us choose y axis to be positive in the upward direction (fig. 6.5). Then for the projections of forces along the y axis we can write:

$$F_l + F_r - mg - Mg = 0 \quad \text{or} \quad F_l = mg + Mg - F_r.$$

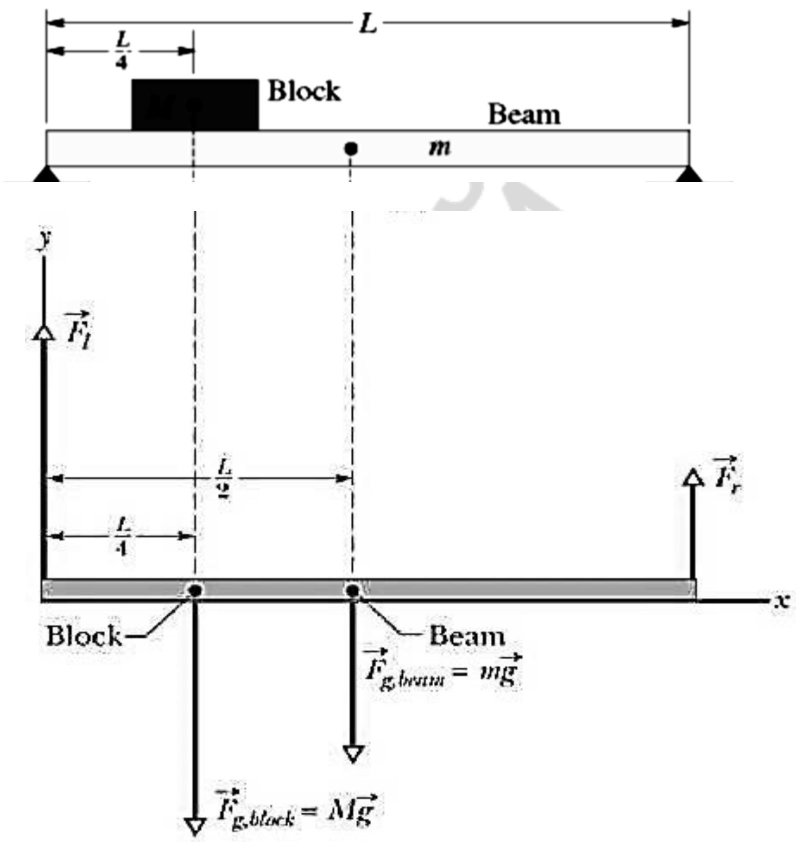


Fig. 6.5. Example 6.2

Then we need to use the Eq. 6.5 (balance of torques). Let us choose the axis O through the left end of the beam so that the torque of the force F_l is equal to zero. Then from the balance of torques equation we obtain:

$$F_r L - mg\frac{L}{2} - Mg\frac{L}{4} = 0.$$

Solving the above equation for F_r yields:

$$F_r = \frac{mg}{2} + \frac{Mg}{4} = \frac{(2m + M)g}{4} = \frac{(2 \cdot 2\ \text{kg} + 3.5\ \text{kg})\ (9.8\ \text{m/s}^2)}{4} = 18.4\ \text{N}.$$

Let us find the remaining unknown force magnitude F_l from the balance of forces equation:

$$F_l = mg + Mg - F_r = (m + M)\cdot g - F_r = (5.5\text{ kg})\,(9.8\text{ m/s}^2) - (18.4\text{ N}) = 35.5\text{ N}.$$

Second solution. This problem can be solved in a different way, applying the balance of torques equation (Eq. 6.5) about two different axes. Choosing first an axis through the left end of the beam, as we did above we find the reaction force at the right end of the beam F_r = 18.4 N.

For an axis passing through the right end of the beam Eq. 6.5 yields

$$-F_l L + mg\frac{L}{2} + Mg\frac{3L}{4} = 0.$$

Solving for F_l, we find

$$F_l = \frac{mg}{2} + \frac{3Mg}{4} = \frac{(2m + 3M)g}{4} = \frac{(2\cdot 2\text{ kg} + 3\cdot 3{,}5\text{ kg})\,(9.8\text{ m/s}^2)}{4} = 35.5\text{ N}.$$

It is in agreement with the previous result. Note that the reaction force magnitudes are not dependent on the beam length, but only on the mass of the beam and the block.

***Example 6.3*. Balance of a horizontal beam.**

A uniform horizontal rod of length L = 5 m rests on two fulcrums, located at distances a = 0.8 m and b = 1.2 m from the left and right ends of the rod, respectively (fig. 6.6). Find the ratio of the reaction forces N_2 and N_1 exerted on the rod by the right and left fulcrums.

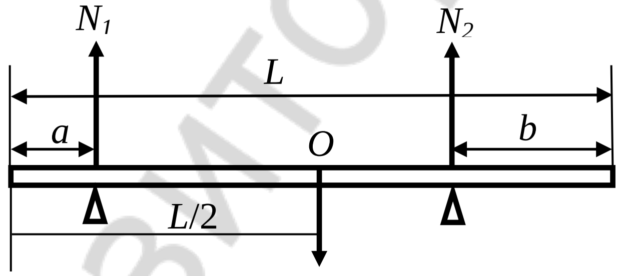


Fig. 6.6. Example 6.3

Solution. A free-body diagram for the rod is shown in fig. 6.6. The forces acting on the rod are as follows: the force of gravity of the rod mg and the reaction forces N_1 and N_2. The problem can be solved most easily by using the torque equation (Eq. 6.5) and by choosing the axis through the point O so that the torque of the force mg is equal to zero.

We calculate torques about the point O. Let $d_1 = \frac{L}{2} - a$ and $d_2 = \frac{L}{2} - b$ are lever arms of the forces N_1 N_2, respectively. Condition for equilibrium of the rod gives:

$$N_1 \cdot d_1 - N_2 \cdot d_2 = 0$$

or $$N_1\left(\frac{L}{2}-a\right)=N_2\left(\frac{L}{2}-b\right).$$

Solving the above equation for N_1/N_2 we obtain:

$$\frac{N_2}{N_1}=\frac{\frac{L}{2}-a}{\frac{L}{2}-b}=\frac{2.5m-0.8m}{2.5m-1.2m}=\frac{1.7m}{1.3m}=1.33.$$

***Example 6.4*. The third class lever. Force exerted by biceps muscle.**

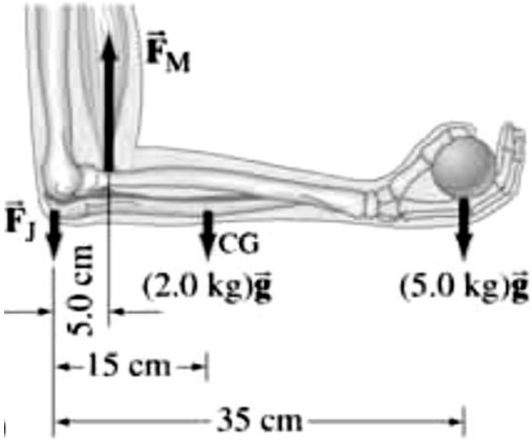


Fig. 6.7. Example 6.4

How much force must the biceps muscle exert when a 5.0-kg ball is held in the hand with the arm horizontal as in fig. 6.7? The biceps muscle is connected to the forearm by a tendon attached 5.0 cm from the elbow joint. Assume that the mass of forearm and hand together is 2.0 kg and their center of gravity (CG) is situated as shown in fig 6.7.

Solution. The free-body diagram for the forearm is shown in fig. 6.7. The forces acting on the lever are as follows the force of gravity of the arm F_a and the ball F_b, the upward force F_M exerted by the muscle, and a force F_J exerted at the joint by the bone in the upper arm (all assumed to act vertically). We wish to find the magnitude of F_M. The problem can be solved most easily by using the torque equation (Eq. 6.5) and by choosing the axis through the joint so that the torque of the force F_J is equal to zero.

We calculate torques about the point where Fj acts. Let d_M, d_a d_b are lever arms of the forces F_M F_a and F_b, respectively. Condition for equilibrium of the lever gives: $$F_M \cdot d_M - F_a \cdot d_a - F_b \cdot d_b = 0.$$

Solving the above equation for F_M and setting $F_a = m_a \cdot g$ and $F_b = m_b \cdot g$ we obtain: $$F_M=\frac{F_a\cdot d_a+F_b\cdot d_b}{d_M}=\frac{(m_a\cdot d_a+m_b\cdot d_b)g}{d_M}.$$

Setting d_M = 5 cm = 0.05 m, d_a = 15 cm = 0.15 m, d_b = 35 cm = 0.35 m; m_a = 2 kg, m_b = 5 kg we calculate the magnitude of the biceps muscle force F_M:

$$F_M=\frac{(m_a\cdot d_a+m_b\cdot d_b)g}{d_M}=\frac{(2\text{ kg})(0.15\text{ m})+(5\text{ kg})(0.35\text{ m})}{0.05\text{ m}}\cdot 9.8\text{ m/s}^2=$$

$$=(41\text{ kg})\ 9.8\text{ m/s}^2\ =402\text{ N}.$$

Note that the force required of the muscle (402 N) is quite large compared to the weight of the object lifted (F_b = 49 N). Indeed, the muscles and joints of the body are generally subjected to quite large forces. The position (distance d_M) at which the biceps muscle is connected to the forearm differs in some extent for different humans. Even small increase in the lever arm of the force exerted by the muscle the can lead to the noticeable increase in the ability of a human to lift weights.

PROBLEMS

1. Cylindrical tube of small diameter with mass $m = 2 \cdot 10^3$ kg is lying on the ground. What the minimum force must be exerted on one of the tube ends to lift it from the ground? (Answer: 9.8 kN).

2. A uniform beam of mass m = 140 kg is suspended horizontally by two vertical ropes. Find the tension in each rope, if the distance from the center of mass of the beam to the ropes are l_1 = 3 m, l_2 = 1 m, respectively. (Answer: 343 N, 1029 N).

3. A uniform horizontal beam of length L = 4 m rests on two fulcrums, located at distances a = 0.8 m and b = 1.2 m from the left and right ends of the beam, respectively. To balance the reaction forces, a load of mass m = 30 kg is suspended at one of the beam ends. Find the mass of the beam. (Answer: 270 kg)

4. A uniform horizontal beam of mass m = 360 kg and length L = 5 m rests on two fulcrums, located at distances a = 1.0 m and b = 1.5 m from the left and right ends of the beam, respectively. A load of some mass is suspended at one of the beam ends in order to provide the reaction forces exerted on the beam by both fulcrums to be equal. Calculate the mass of the load. (Answer: 40 kg)

7. FLUID MECHANICS

In this Chapter we consider the materials that are very deformable and can flow. Such "fluids" include liquids and gases. We concentrate on the main points of hydrostatics — the branch of fluid mechanics that is related to the study of fluids at rest in equilibrium state. Hydrodynamics is the study of fluids in motion.

The three common phases, or states, of matter are solid, liquid, and gas. We can distinguish these three phases as follows. A solid maintains a fixed shape and a fixed size. A liquid preserves its volume, but does not maintain a fixed shape, it takes on the shape of its container. A gas has neither a fixed shape nor a fixed volume, it will expand to fill its container. Since liquids and gases do not maintain a fixed shape, they both have ability to flow; they are thus often referred to fluids.

7.1. DENSITY AND PRESSURE

An important property of any material is its ***density*** **ρ**, defined as its mass per unit volume. If a homogeneous material with a mass ***m*** has volume ***V***, the density is

$$\boldsymbol{\rho = \frac{m}{V}}. \tag{7.1}$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. Density is a scalar quantity; its SI unit is the kilogram per cubic meter (kg/m^3).

NOTE. *The density of a gas varies considerably with pressure, but the density of a liquid does not; i. e. liquids are not compressible*.

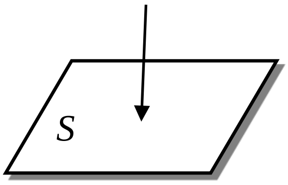


Fig. 7.1. A force acting perpendicular to the surface of area *S*

Consider the force ***F*** acting perpendicular to the surface with area ***S*** (normal force) (fig. 7.1). Pressure ***p*** is defined as magnitude of the normal force per unit area:

$$\boldsymbol{p = \frac{F}{S}}. \tag{7.2}$$

Pressure is a scalar. The SI unit of pressure is pascal (Pa) (after the name of Blaise Pascal) (1 Pa = N/m^2).

The pascal is related to some other common (non-SI) pressure units as follows:

$$1 \text{ atm} = 1.013 \cdot 10^5 \text{ Pa} = 760 \text{ torr}.$$

The *atmosphere* (abbreviated atm) is the approximate average pressure of the atmosphere at sea level. The *torr* (named for Evangelista Torricelli, who invented the mercury barometer) is the pressure exerted by a mercury column of 1 millimeter high (1 torr =1 mm Hg = 133.3 Pa).

7.2. PASCAL'S PRINCIPLE

Solids and fluids transmit forces differently. When a force is applied to one section of a solid, this force is transmitted to the other parts of the solid in the same direction. Because of a fluid's ability to flow, it transmits a force uniformly in all directions. Therefore the additional pressure at any point in a fluid at rest is the same in all directions. This principle is known as ***Pascal's principle*** (after the name of French scientist Blaise Pascal).

Pascal's principle states: *the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of its container.*

From Pascal's principle it follows that the pressure at a point in a fluid in static equilibrium depends on the depth of that point but not on any horizontal dimension of the fluid or its container.

Let us find quantitatively how the pressure in a liquid of uniform density varies with depth. Consider a point at a depth ***h*** below the surface of the liquid, as shown in fig. 7.2. The pressure at this depth ***h*** is due to the weight of the column of liquid above it. Thus the force due to the weight of liquid acting on the area ***S*** is defined as

$$\boldsymbol{F = mg = (\rho \cdot V) \cdot g = \rho \cdot S \cdot h \cdot g}, \qquad (7.3)$$

where ***V = S·h*** is the volume of the column of liquid, **ρ** is the density of the liquid, and ***g*** is the acceleration due to gravity. The pressure ***p*** due to the weight of liquid is then

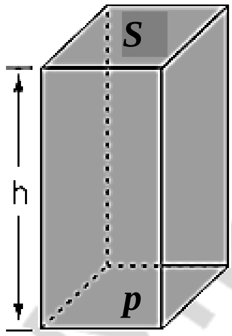


Fig. 7.2. Hydrostatic pressure at the depth ***h***

$$p = \frac{F}{S} = \frac{\rho S h g}{S} = \rho g h. \qquad (7.4)$$

This pressure is called hydrostatic.

NOTE. *The area S doesn't affect the hydrostatic pressure. The fluid pressure is directly proportional to the density of the liquid and to the depth within the liquid. In general, the pressure at equal depths within a uniform liquid is the same. The shape of the container does not matter* (fig. 7.3).

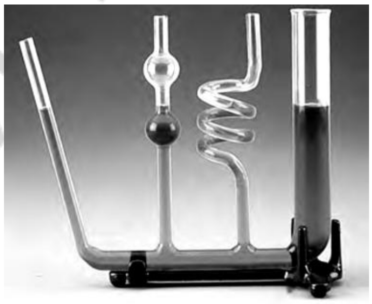


Fig. 7.3. The pressure in a fluid is the same for all points at the same level

In case of a liquid in an open container (e. g. water in a glass) there is a free surface at the top exposed to the atmosphere. Then the pressure at a depth ***h*** in the fluid is

$$\boldsymbol{p = p_0 + \rho g h}, \qquad (7.5)$$

where $\boldsymbol{p_0}$ is the atmospheric pressure at the top surface.

***Example 7.1*. Pressure. Pascal's Principle.**

Determine the total force and the pressure on the bottom of a swimming pool with transverse sizes 25.0 m by 12.5 m and depth 2 m (the density of water $\rho = 10^3$ kg/m^3). What will be the pressure on the wall of the pool near the bottom?

Solution. For solving a problem we use Eqs. 7.2 and 7.4. The surface area of the swimming pool bottom is $S = a \cdot b$, where a and b are the width and the length of the pool.

Setting $\rho = 10^3$ kg/m^3, $h = 2$ m and g = 9.8 m/s^2 in Eq. 7.4, we obtain hydrostatic pressure on the bottom of the swimming pool

$$p = \rho g h = (10^3\ \text{kg/m}^3)\ (9.8\ \text{m/s}^2) \cdot (2\ \text{m}) = 1.96 \cdot 10^4\ \text{Pa}.$$

From Eq.7.2 it follows that the total force F acting on the pool bottom is $F = p \cdot S = p \cdot a \cdot b$. Setting $a = 12.5$ m, $b = 25.0$ m, we obtain:

$$F = (19.6 \cdot 10^3\ \text{Pa}) \cdot (12.5\ \text{m}) \cdot (25.0\ \text{m}) = 6.125 \cdot 10^6\ \text{N}.$$

Pressure is $p = 19.6 \cdot 10^3$ Pa, the force is $F = 6.125 \cdot 10^6$ N.

According to Pascal's principle the pressure is the same for all points at equal depth, thus the pressure on the wall of the pool near the bottom is $p = 19.6 \cdot 10^3$ Pa.

7.3. ARCHIMEDES' PRINCIPLE AND BUOYANCY

7.3.1. ARCHIMEDES' PRINCIPLE

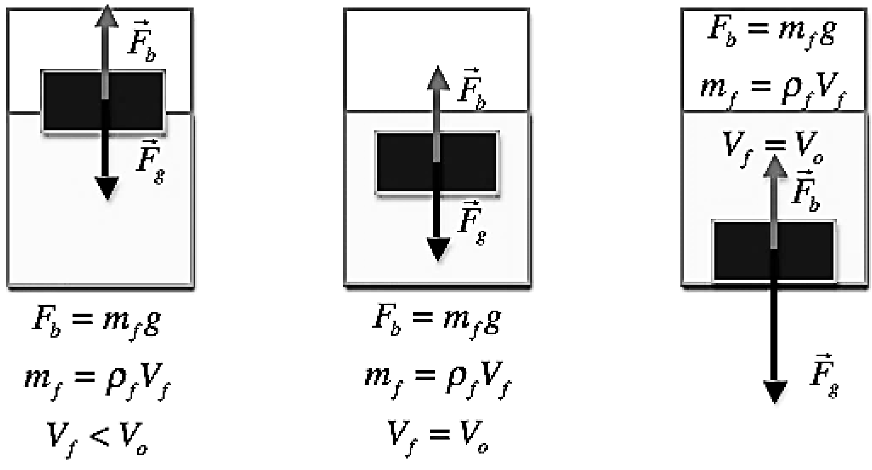
Archimedes' principle states: *when a body is completely or partially submerged in a fluid, a buoyant force* (***F***$\boldsymbol{F_b}$) *from the surrounding fluid acts on the body* (fig. 7.4). The force is directed upward and has a magnitude equal to the weight of the fluid that has been displaced by the body[1]:

$$\boldsymbol{F_b = m_f \cdot g = \rho_f \cdot V_f \cdot g}, \qquad (7.6)$$

where $\boldsymbol{m_f}$ is the mass of the fluid displaced by the body; ρ_f is the fluid density, $\boldsymbol{V_f}$ is the submerged volume of a body, $\boldsymbol{g}$ is an acceleration due to gravity.

The buoyant force occurs because the pressure in a fluid increases with depth.

Objects submerged in a fluid appear to weigh less than they do when outside the fluid. For example, a large rock that you would have difficulty lifting off the ground can often be easily lifted from the bottom of a stream. When the rock breaks through the surface of the water, it suddenly seems to be much heavier. Many objects, such as wood, float on the surface of water. These are examples of buoyancy (fig. 7.4). In each example, the force of gravity is acting downward. But in addition, an upward buoyant force is exerted by the liquid.



a ***b*** ***c***

Fig. 7.4. Examples of buoyancy:
a — a body is partially submerged; *b* — a body is completely submerged; *c* — a body sinks ($\boldsymbol{V_0}$ is the volume a body, $\boldsymbol{V_f}$ is the submerged volume of a body)

[1] By "fluid displaced", we mean a volume of fluid equal to the submerged volume of the object (or that part of the object that is submerged).

The weight of a body on which a buoyant force acts is given by the following expression:

$$W = mg - F_b = (m - \rho_f \cdot V_f) \cdot g. \qquad (7.7)$$

The net downward force acting on a body of a mass m submerged in a fluid is the difference between the gravitational force and the buoyant force:

$$F = F_g - F_b = mg - \rho_f \cdot V_f \cdot g = \rho \cdot Vg - \rho_f \cdot V_f \cdot g, \qquad (7.8)$$

where ρ and ρ_f are the body and fluid density, correspondently; V is the body volume and V_f is the fraction of a body volume which is submerged.

From Eq. 7.8 it follows that whether a body sinks or floats in liquid depends on its density. If its density is greater than that of liquid **($\rho > \rho_f$)**, then $F_g > F_b$ and a body sinks. When the magnitude F_b of the (upward) buoyant force acting on the body is equal to the magnitude F_g of the (downward) gravitational force, a body floats in a fluid.

7.3.2. CONDITION FOR FLOTATION

Using Eq. 7.8 we obtain a condition at which a body floats at the surface of a liquid:

$$mg = \rho_f \cdot V_f \cdot g \quad \text{or} \quad \rho \cdot Vg = \rho_f \cdot V_f \cdot g. \qquad (7.9)$$

From the Eq. 7.9 a condition for flotation of bodies can be expressed as follows:

$$\frac{\rho}{\rho_f} = \frac{V_f}{V}. \qquad (7.10)$$

***Example 7.2*. Buoyancy.**

A wood block of density 800 kg/m^3 floats in a liquid of density 1200 kg/m^3. The block has height h = 6 cm. By what depth is the block submerged?

Solution. The block floats when the buoyant force is equal to the gravitational force (fig. 7.5). To find depth h we use Eq. 7.10. The volume of the displaced liquid is $V = h \cdot S$, where S is the area of the block surface submerged in the liquid.

From Eq. (7.10) it follows $\frac{\rho}{\rho_f} = \frac{V_f}{V} = \frac{h_f \cdot S}{h \cdot S} = \frac{h_f}{h}$.

Fig. 7.5. Example 7.2. Forces acting on a wood block submerged in a liquid

Then the depth h by which the block submerged is defined by the following equation:

$$h_f = \frac{\rho}{\rho_f} \cdot h.$$

Setting ρ_f = 1200 kg/m^3, ρ = 800 kg/m^3, h = 6 cm = 0.06 m, we obtain h_f:

$$h_f = \frac{\rho}{\rho_f} \cdot h = \frac{0.8 \cdot 10^3 \text{ kg/m}^3}{1.2 \cdot 10^3 \text{ kg/m}^3} \cdot 6 \cdot 10^{-2} \text{ m} = 4 \cdot 10^{-2} \text{ m} = 0.04 \text{ m}.$$

The depth by which the block is submerged is 4 cm.

PROBLEMS

1. Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What is the pressure at the bottom of the tank? (Answer: $2.2 \cdot 10^5$ Pa)

2. What is the difference in hydrostatic blood pressure (mm Hg) between the top of the head and the bottom of the feet of 1.70 m tall person standing vertically? (the density of blood $\rho_{blood} = 1.06 \cdot 10^3$ kg/m^3). (Answer: 133 mm Hg)

3. If the force on the tympanic membrane (eardrum) increases by about 1.5 N above the force from atmospheric pressure, the membrane can be damaged. When a man goes scuba diving in the sea, below what depth could damage to his eardrum start to occur? The eardrum is typically 8.2 mm in diameter (the density of seawater $\rho = 1.03 \cdot 10^3$ kg/m^3). (Answer: 2.8 m)

4. A manometer tube is partially filled with water. Both arms of the tube are open to the air. Oil (which does not mix with water) is poured into the left arm of the tube. Find the ratio between the heights of oil and water columns in the left and right arms correspondently over the level of oil–water interface ($\rho_{oil} = 0.86 \cdot 10^3$ kg/m^3) (Answer: 1.16).

5. A geologist finds that a Moon rock with an actual mass 9 kg has a mass of 6 kg when completely submerged in water. What is the density of the rock? (Answer: $3 \cdot 10^3$ kg/m^3)

TESTS

1. SI units of pressure is
a) torr b) atm c) Pa d) bar

2. Hydrostatic pressure depends on liquid
a) density b) viscosity c) volume d) none of the above

3. A body floats in liquid when
a) $F_g > F_b$ b) $F_g < F_b$ c) $F_g = F_b$ d) none of the above

4. What fraction of a piece of iron will be submerged when it floats in mercury? ($\rho_{iron} = 7.8 \cdot 10^3$ kg/m^3, $\rho_{mercury} = 13.6 \cdot 10^3$ kg/m^3)
a) 0.28 b) 0.42 c) 0.57 d) 0.65

8. FUNDAMENTALS OF KINETIC THEORY OF GASES

8.1. ASSUMPTIONS OF KINETIC MOLECULAR THEORY OF GASES

The kinetic theory of gases is the study of the structure and properties of substance which describes a gas as a large number of small particles (atoms or molecules) in permanent random motion. There are following assumptions of this model:

- A gas consists of large number of small particles (atoms or molecules), which are in permanent random motion. The random motion means that any molecule can move in any direction with different speed at any moment.
- The molecules undergo elastic collisions with each other. The interaction forces between gas molecules are negligible except during a collision, because the average separation between particles is great compared with their dimensions.
- The molecules undergo elastic collisions with the walls of their container and produce a pressure on the walls.

This properties of gas molecules explain many phenomena, as well known Brownian motion, diffusion and etc.

A ***diffusion*** is the movement of atoms and molecules from a region of high concentration to a region of low concentration. For example, some particles are dissolved in a glass of water. At first, the particles are all near one side of the glass. If the particles all randomly move around — diffuse — in the water, then the particles will eventually become distributed randomly and uniformly (fig. 8.1).

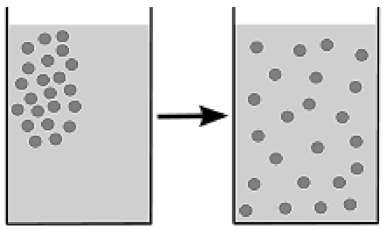


Fig. 8.1. Diffusion

8.2. AMOUNT OF SUBSTANCE, MOLAR MASS

Amount of substance (chemical amount) $\boldsymbol{n}$ is a quantity that measures the amount of ensemble of atoms, molecules or other particles. The SI unit for amount of substance is the mole. The ***mole*** is defined as the amount of substance that contains the Avogadro's number ($\boldsymbol{N_A = 6.02 \cdot 10^{23}}$) of its elementary particles (atoms or molecules). Such amount of atoms is contained in 12 g of the isotope carbon-12.

Amount of substance in moles can be determined as:

$$n = \frac{N}{N_A}, \qquad (8.1)$$

where N is number of molecules in substance.

Molar mass M is a mass of 1 mole of a given substance, its physical characteristic equal to the mass ***m*** of substance per its amount of substance ***n***:

$$M = \frac{m}{n}. \tag{8.2}$$

The SI unit for molar mass is ***kg/mol***. However, for both practical and historical reasons, molar masses are almost always measured in ***g/mol***.

Molecular mass can be determined from the following relation:

$$m_0 = \frac{M}{N_A}. \tag{8.3}$$

8.3. IDEAL GAS. GAS PRESSURE

An **ideal gas** is a theoretical gas composed of many randomly moving point particles that do not interact except when they collide elastically.

At normal conditions such as standard temperature and pressure, most real gases such as hydrogen, helium, neon, nitrogen, oxygen behave qualitatively like an ideal gas.

As gas particles are constantly moving, they are also constantly colliding with the walls of their container and provide the forces pushing this walls (fig. 8.2). The gas pressure is the sum of these forces ***F*** divided by the area ***S*** of the container wall:

$$p = \frac{F}{S}. \tag{8.4}$$

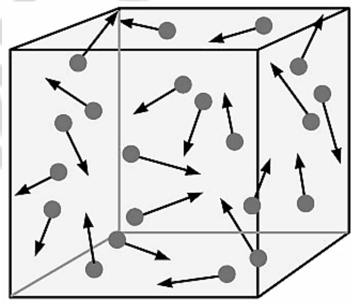


Fig. 8.2. Gas molecules colliding with container walls

The molecule-kinetic theory of ideal gas shows that the gas pressure on the container walls is determined by formula:

$$p = \frac{1}{3} c m_0 \overline{\upsilon^2}, \tag{8.5}$$

where $c = \frac{N}{V}$ is concentration of gas molecules (a number of molecules per unit volume), m_0 is mass of molecule, $\sqrt{\overline{\upsilon^2}}$ is the mean square speed of the molecule.

As an average kinetic energy of one molecule is $\overline{E}=\frac{m_0\overline{\upsilon^2}}{2}$, the gas pressure depends on the kinetic energy of gas molecules:

$$p=\frac{2}{3}c\frac{m_0\overline{\upsilon^2}}{2}=\frac{2}{3}c\overline{E}. \tag{8.6}$$

The equation (8.6) is a very important in the kinetic theory because it relates pressure, a macroscopic property, to the average kinetic energy per molecule which is a microscopic property. The SI unit of pressure is a ***pascal*** (Pa) which is a newton per square meter (1 Pa = 1 N/m^2). Other common units of pressure are the ***atmosphere*** (atm) and millimeter of mercury (mmHg):

1 atm = 760 mmHg = 101325 Pa ≈ 10^5 Pa.

8.4. TEMPERATURE AS A MEASURE OF KINETIC ENERGY OF MOLECULES

From the equation (8.6) we can find the average kinetic energy of gas molecules as

$$\overline{E}=\frac{3pV}{2N}. \tag{8.7}$$

The special experiments have shown that at any constant temperature the value $\left(\frac{pV}{N}\right)$ is the same for any gas and it is proportional to the gas temperature. English physicist William Thomsom, 1st Baron Kelvin suggested the ***absolute temperature scale*** **T** in which

$$\left(\frac{pV}{N}\right)=kT. \tag{8.8}$$

and k is called Boltzmann constant ($k = 1.38 \cdot 10^{-23}$ J/K).

Zero of ***Kelvin temperature*** scale defines the absolute zero, a hypothetical temperature at which all molecular movement stops and gas pressure drops to zero. All actual temperatures are above absolute zero.

The average kinetic energy of molecules of the ***monoatomic gas*** depends on the absolute temperature according to the formula:

$$\overline{E}=\frac{m_0\overline{\upsilon^2}}{2}=\frac{3}{2}kT. \tag{8.9}$$

From (8.9) the ***mean square speed*** of gas molecules is:

$$\upsilon_{mss}=\sqrt{\overline{\upsilon^2}}=\sqrt{\frac{3kT}{m_0}}=\sqrt{\frac{3RT}{M}}. \tag{8.10}$$

where $R = N_A \cdot k = 8.31\ \frac{\text{J}}{\text{K}\cdot\text{mol}}$ is the constant called the ***universal gas constant***, $M = m_0N_A$ is molar mass.

Therefore, ideal gas pressure in absolute temperature scale is:

$$\boldsymbol{p = ckT.} \qquad (8.11)$$

Usually we use a ***Celsius temperature scale*** (fig. 8.3). According to this scale the temperature difference between the reference temperatures of the freezing and boiling points of water is divided into 100 degrees. The freezing point is taken as 0 Celsius degrees and the boiling point as 100 Celsius degrees.

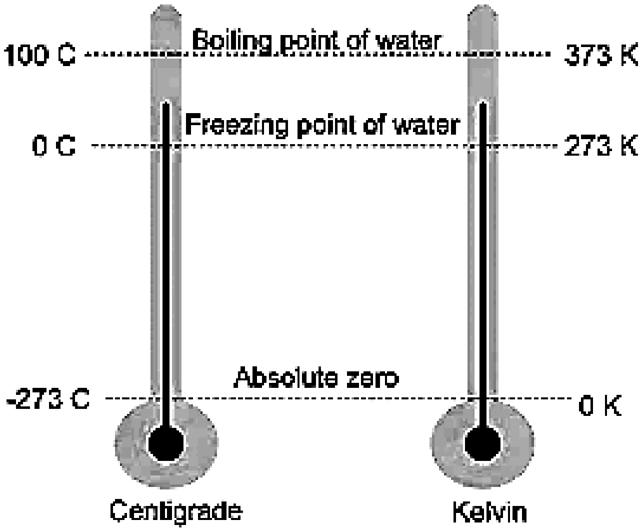


Fig. 8.3. Relation between Celsius and Kelvin temperature scales

The intervals for the Celsius scale and for Kelvin scale are the same **1 K = 1 °C**, but the absolute zero **0 K = –273.15 °C**.

Thus, the conversion between these temperatures is:

$$\boldsymbol{T_K \approx T_C + 273.} \qquad (8.12)$$

8.5. IDEAL GAS LAW

An ideal gas state can be characterized by three main values: pressure ***p***, volume ***V***, and absolute temperature ***T***. The ***Ideal Gas Law*** (the equation of state for an ideal gas) is general relation between these variables at fixed quantity of gas. As an ideal gas pressure (8.11) is ***p = ckT***, where $\boldsymbol{c = \frac{N}{V} = \frac{nN_A}{V}}$, we can write:

$$\boldsymbol{p = \frac{n}{V} N_A kT = \frac{n}{V} RT,} \qquad (8.13)$$

The most frequently introduced form of this equation is:

$$\boldsymbol{pV = nRT} \quad \text{or} \quad \boldsymbol{pV =} \frac{m}{M} RT, \qquad (8.14)$$

where ***m*** is mass, ***M*** is molar mass and ***n*** is amount of gas moles.

According to the equation of state, one of the thermodynamic variables ***p***, ***V*** or ***T*** may always be expressed as the function of the other two values.

Example 8.1.

A helium balloon, assumed to be a perfect sphere, has a radius of 18 cm. At room temperature (20 °C), its internal pressure is 1.05 atm. Find the number of moles of helium in the balloon and the mass of helium needed to inflate the balloon to these values.

***Solution*:**

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.18\text{ m})^3 = 0.0244\text{ m}^3.$

$P = 1.05\text{ atm} = 1.064 \cdot 10^5\text{ N/m}^2.$

$T = 20\text{ °C} = (20 + 273)\text{ K} = 293\text{ K}.$

$R = 8.31\text{ J/(mol·K)}.$

Thus $n = \frac{PV}{RT} = \frac{(1.064 \cdot 10^5\text{ N/m}^2)(0.0244\text{ m}^3)}{(8.31\text{ J/mol}\cdot\text{K})(293\text{ K})} = 1.066\text{ mol}.$

The atomic mass of helium M = 4 g/mol, so gas mass is

$$m = n \cdot M = 1.66\text{ mol} \cdot \frac{4\text{ g}}{\text{mol}} = 4.26\text{ g} = 4.26 \cdot 10^{-3}\text{ kg}.$$

8.6. ISOMETRIC PROCESSES

Isometric processes are thermodynamic processes during which the amount of substance and one of the state variables — pressure, volume or temperature — remain unchanged.

1. Isothermal process is carried out with a fixed amount of gas at constant temperature. In this case, the product of an ideal gas pressure and volume is always constant (***Boyle's law***):

$$\boldsymbol{PV = \text{const} = nRT}$$

or

$$\boldsymbol{p_1V_1 = p_2V_2}, \tag{8.15}$$

where $\boldsymbol{n = \text{const}}$, $\boldsymbol{T = \text{const}}$.

A plot of $\boldsymbol{p}$ versus $\boldsymbol{V}$ at constant temperature for an ideal gas is a hyperbolic curve called an ***isotherm*** (fig. 8.4). Each point on the curve represents the state of the system at a given moment — that is, at pressure $\boldsymbol{p}$ and volume $\boldsymbol{V}$. At a lower temperature another isothermal process is represented by a lower curve A′B′ (the product $\boldsymbol{pV}$ is less when $\boldsymbol{T}$ is less).

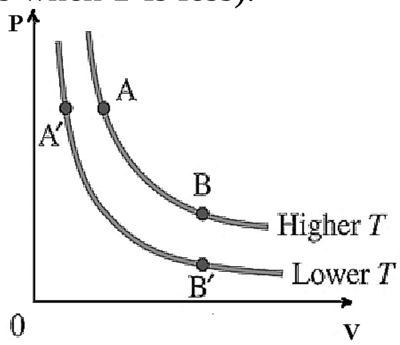


Fig. 8.4. Isotherms for an ideal gas at two different temperatures

2. At ***isobaric process*** the pressure of a fixed amount of gas is kept constant. During the isobaric process the gas volume is proportional to gas temperature:

$$\boldsymbol{V = \frac{nR}{p}T = \mathrm{const} \cdot T,} \tag{8.16}$$

where ***n*** **= const**, ***p*** **= const**.

It is known as ***Charles' law***:

$$\boldsymbol{\frac{V}{T} = \mathrm{const} \quad \mathrm{or} \quad \frac{V_1}{T_1} = \frac{V_2}{T_2}.} \tag{8.17}$$

This process is represented by a straight line on the volume-temperature diagram called ***isobar*** and coming out of the origin of coordinates (fig. 8.5).

3. At ***isochoric process*** the gas volume does not change. The pressure of gas of fixed mass and fixed volume is directly proportional to the absolute temperature of gas. This law is expressed mathematically as:

$$\boldsymbol{p = \frac{nR}{V}T = \mathrm{const} \cdot T,} \tag{8.18}$$

where ***n*** **= const**, ***V*** **= const**, or as ***Gay-Lussac's law***:

$$\boldsymbol{\frac{p_1}{T_1} = \frac{p_2}{T_2}.} \tag{8.19}$$

On pressure-temperature diagram of the isochoric process is represented by a straight line called ***isochor*** (fig. 8.6).

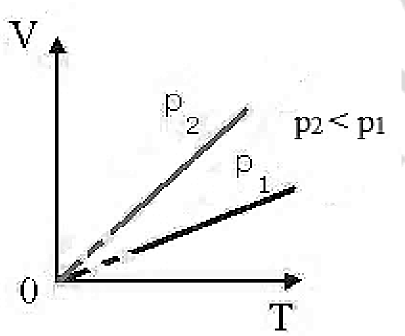


Fig. 8.5. VT-diagram of isobaric process

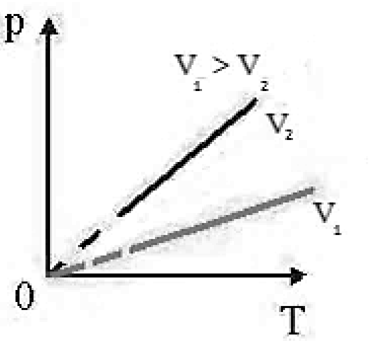


Fig. 8.6. pT-diagram of isochoric process

Example 8.2.

The gas pressure in cylinder is 4.40 kPa at 25°C. At what temperature in °C will it reach a pressure of 6.50 kPa?

Solution. Since a cylinder volume is constant, $V_1 = V_2$, then

$$\frac{P_1}{T_1} = \frac{P_2}{T_2},$$

$$T_2 = \frac{T_1 P_2}{P_1} = \frac{298\ \mathrm{K} \cdot 6.50\ \mathrm{Pa}}{4.40\ \mathrm{kPa}} = 440\ \mathrm{K} = 167\ ^{\circ}\mathrm{C}.$$

TESTS

1. According to the kinetic theory, the velocity of molecules increases with the:

a) rise in temperature;
b) fall in temperature;
c) neither increases nor decreases.

2. According to the kinetic theory, the collision between the molecules of gas are:

a) perfectly inelastic;
b) partially elastic;
c) perfectly elastic;
d) none of the above.

3. According to the gas kinetic theory the molecules:

a) repel each other;
b) collide with each other elastically;
c) move with uniform velocity;
d) are massless particles.

4. Brownian motion is:

a) discontinuous;
b) not random;
c) regular;
d) due to molecular collision.

5. According to kinetic theory of a gas the kinetic energy of a gas is:

a) proportional to the square root of its temperature;
b) independent of its absolute temperature;
c) proportional to its absolute temperature;
d) proportional to cube of absolute temperature.

6. Absolute zero is the temperature at which:

a) all molecular motion ceases;
b) some molecules are at rest;
c) none of the above.

7. The absolute zero is expressed as:

a) 273 ºC; b) –273 ºC; c) –273 K; d) 373 ºC.

8. A gas is at one atmosphere. To what pressure it should be subjected at constant temperature so as to have 1/4th of its initial volume.

a) $^1/_4$ atm; b) 2 atm; c) 3 atm; d) 4 atm.

9. If the temperature of air is increased from 20 ºC to 200 ºC, the increase in kinetic energy will be:

a) 1.6 times; b) 2 times; c) 3.4 times; d) 10 times.

10. The temperature at which the kinetic energy of the gas will be half of the kinetic energy at room temperature of 27 ºC is:

a) 13.5 ºC; b) –27 ºC; c) 123 ºC; d) –123 ºC.

11. An ideal gas at 27ºC is heated at constant pressure so as to double its volume. The temperature of the gas will be:

a) 54 ºC; b) 327 ºC; c) 450 ºC; d) 600 ºC.

12. In which process *PV*-diagram is a straight line parallel to the volume axis?

a) isothermal; b) isochoric;
c) isobaric; d) none of these.

13. Isothermal relation for 1g of a gas is:

a) $PV = RT$; b) PV = constant;
c) P/T = constant; d) V/T = constant.

14. An ideal gas is expanded isothermally, its temperature will:

a) increase; b) decrease;
c) remain the same; d) become zero.

15. Gas occupies 100 ml volume at 10^4 Pa pressure. If during isothermal process the pressure is changed to 10^3 Pa the volume of the gas will be:

a) 10 ml; b) 50 ml; c) 200 ml; d) 1000 ml.

PROBLEMS

16. What is the pressure inside 38 L container holding 105 kg of argon gas at 20 °C? (Answer: 168 MPa)

17. If 61.5 L of oxygen at 18°C and an absolute pressure of 245 kPa are compressed to 48.8 L and at the same time the temperature is raised to 56 °C, what will the new pressure be? (Answer: 349 kPa)

9. THERMAL PHENOMENA. BASICS OF THERMODYNAMICS

Thermodynamic is the theory of thermal phenomena which is not considered atomic and molecular structure of bodies.

A thermodynamic system is the content of a macroscopic volume in space, along with its walls and surroundings; it undergoes thermodynamic processes according to the principles of thermodynamics. This system can be described by thermodynamic variables such as temperature, internal energy and pressure.

9.1. INTERNAL ENERGY. WORK OF GAS. FIRST LAW OF THERMODYNAMICS

The total sum of the energies of all molecules in an object is called its ***internal energy***. It includes total kinetic energy of the molecule's motion and total potential energy of their interactions. Energy can be transferred between a molecule system and its surroundings in two ways:

a) ***work W*** done by the system;

b) ***heat Q transfer***, which occurs between the bodies with different temperature.

The work produced by gas is equal to

$$W = F_{gas} \cdot \Delta l \cdot \cos\alpha = p \cdot S \cdot \Delta l \cdot \cos\alpha = p \cdot \Delta V. \tag{9.1}$$

If the gas expands, $\Delta V \equiv V_2 - V_1 > 0$ and gas work is positive $W > 0$. The work of external force ($\vec{F}_{ext} = -\vec{F}_{gas}$) is negative in this case. Vice versa if the gas is compressed by the external force the $\Delta V \equiv V_2 - V_1 < 0$ and gas work is negative $W < 0$, but the work of external force is positive.

The heat Q is transferred from body of higher temperature to the body of lower temperature. The transferred heat $Q > 0$ if thermal energy enters the body and $Q < 0$ when this energy leaves it. The SI unit of heat quantity is ***Joule***.

The First law of thermodynamics: the changing of the internal energy ΔU of molecular system is due to heat Q transfer and to the work W done by the system:

$$\Delta U = Q - W. \tag{9.2}$$

In this formula $Q > 0$ when thermal energy enters the system and $Q < 0$ when this energy leaves the system.

The ***First law of thermodynamics*** (9.2) may be written in the form**:**

$$Q = \Delta U + W. \tag{9.3}$$

The ***internal energy*** of n moles of an ideal monatomic (one atom per molecule) gas is equal to

$$U = \frac{3}{2} nRT. \tag{9.4}$$

Thus, the internal energy of an ideal gas depends only on temperature and the number of moles of gas.

The changing of internal energy ΔU of an ideal monatomic gas is equal to:

$$\Delta U = \frac{3}{2} nR \cdot \Delta T. \quad (9.5)$$

9.2. FIRST LAW OF THERMODYNAMICS AT DIFFERENT PROCESSES

At ***isothermal process*** the T is constant and internal energy of gas doesn't change: $\Delta T = 0$ and $\Delta U = 0$. First Law is:

$$Q = W.$$

That means that all heat energy transfers to a work.

At ***isobaric process*** gas pressure is fixed but the gas volume and temperature may changes and First Law is:

$$Q = \Delta U + W,$$

one part of transferred heat goes to doing a work by system and the another part goes to increasing the internal energy of gas.

At this process $W = p \cdot \Delta V = nR \cdot \Delta T$.

For ideal monoatomic gas $\Delta U = 1.5 \cdot nR \cdot \Delta T$, so at this case

$$Q = 2.5nR \cdot \Delta T.$$

At ***isochoric process*** gas volume is fixed and gas doesn't make a work: $W = p \cdot \Delta V = 0.$ So the First Law in this case is $Q = \Delta U$, and for monoatomic gas $Q = \Delta U = 1.5nR \cdot \Delta T$.

All heat transfers to internal energy. It is the best process for body heating.

There is a process, when the heating is absent: $Q = 0$. This process is called "***adiabatic process***". In this case First Thermodynamics Law has a form:

$$\Delta U + W = 0 \quad \text{or} \quad \Delta U = -W.$$

It means that the gas internal energy decreases (and gas temperature decreases too) if gas adiabatically extends ($W > 0$) and vice versa, gas temperature and internal energy increases if gas is adiabatically compressed ($W < 0$).

Example 9.1.

2500 J of heat is added to a system, and 1800 J of work is done on the system. What is the changing of internal energy of the system?

Solution. The heat added to the system is $Q = 2500$ J. The work W done by the system is –1800 J. Why the minus sign? Because 1800 J done on the system (as given) equals –1800 J done by the system. Hence:

$$\Delta U = Q - W = 2500\text{ J} - (-1800\text{ J}) = 2500\text{ J} + 1800\text{ J} = 4300\text{ J}.$$

9.3. HEAT TRANSFER, TYPES OF HEAT TRANSFER

The transfer of heat normally occurs from a higher temperature object to a lower temperature object. Heat transfer changes the internal energy of both objects involved. There are three different ways to transfer energy from one part of system to another: conduction, convection, and radiation. In practical situations, any two or all three may be operating at the same time.

Conduction is heat transfer by means of molecular agitation within a material without any motion of the material as a whole. If one end of a metal rod is at a higher temperature, then energy will be transferred along the rod toward the colder end. It happens due to the higher speed particles will collide with the slower ones with a transfer of energy to them.

Convection is heat transfer by mass motion of a fluid such as air or water when the heated fluid moves away from the source of heat, carrying energy with it. Convection above a hot surface occurs because hot air expands, becomes less dense, and rises. Hot water is likewise less dense than cold water and rises, causing convection currents which transport energy.

Thermal radiation is energy transfer by the emission of electromagnetic waves which carry out energy away from the emitting object. All life on Earth depends on radiation from the Sun, which consists of visible light plus many other wavelengths that the eye is not sensitive to, including much of the infrared radiation.

9.4. AMOUNT OF HEAT. SPECIFIC HEAT

The ***amount of heat*** $\boldsymbol{Q}$ required to change the temperature of a given material is proportional to the mass $\boldsymbol{m}$ of the material and to the temperature change $\boldsymbol{\Delta T}$. It can be expressed by the equation:

$$\boldsymbol{Q = cm\Delta T}, \tag{9.6}$$

where $\boldsymbol{c}$ is a quantity characteristic of the material which is called a ***specific heat***.

If $\boldsymbol{\Delta T = T_2 - T_1} > 0$, the $\boldsymbol{Q > 0}$ too (process of body heating), if $\boldsymbol{\Delta T = T_2 - T_1} < 0$ than $\boldsymbol{Q < 0}$ (process of body cooling).

The ***specific heat c*** is the amount of heat per unit mass required to raise the temperature by one Kelvin. Specific heat is essentially a measure of how thermally insensitive a substance is to the addition of energy. The greater a material's specific heat, the more energy must be added to a given mass of the material to cause a given temperature change. The SI unit of specific heat is **J/kg·K**.

9.5. PHASE CHANGES

A ***phase change*** is the transformation of a thermodynamic system from one phase or state of matter to another one by heat transfer. The term is most commonly used to describe transitions between solid, liquid and gaseous states of matter. The graph below (fig. 9.1) shows the matter temperature T changes with the heat Q transfer.

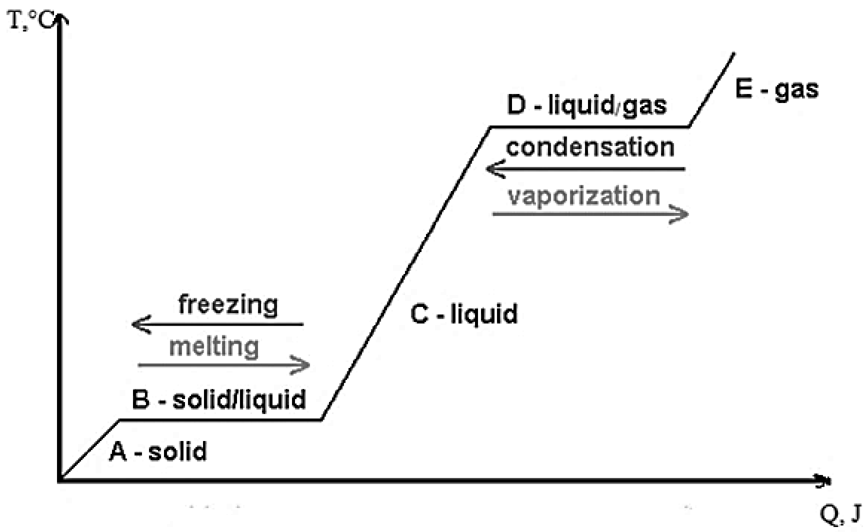


Fig. 9.1. Change temperature of a substance during the heating

Melting is a physical process that results in the phase change from solid to liquid. Inverse process, the transition from liquid to solid, is called ***crystallization***. The ***melting point*** is the temperature at which state of a substance changes from solid to liquid. At the melting point the solid phase and liquid one exist in a heat equilibrium. At the melting point, although the substance is still being heated, there is a time when the temperature does not change and the graph is horizontal. During this time, all the extra heat which is being added goes to overcome the force of attraction between the particles of the solid as it turns into a liquid.

The heat involved in a change of solid-liquid phase depends on the total mass of the substance $\boldsymbol{m}$ as:

$$\boldsymbol{Q = \lambda m}, \qquad (9.7)$$

where $\boldsymbol{\lambda}$ is the ***heat of fusion***.

Heat of fusion is the heat required to change 1.0 kg of a substance from the solid to the liquid state at melting temperature. The heat of fusion of water is 340 kJ/kg.

At any phase of the matter state the amount of heat $\boldsymbol{Q}$ required to change the temperature of the material is expressed by the equation (9.6):

$$\boldsymbol{Q_i = c_i\, m \cdot \Delta T},$$

where $\boldsymbol{c_i}$ is a ***specific heat*** of the corresponding state of a matter.

Vaporization is a phase transition from the liquid phase to vapour. If conditions allow the formation of vapour bubbles within a liquid, the vaporization process is called boiling. ***Boiling*** is a phase transition from the liquid phase to gas phase that occurs at or above the boiling temperature. Direct conversion from solid to vapour is called ***sublimation***.

At the ***boiling point*** bubbles of vapour are formed within the whole liquid volume. The boiling point of a liquid depends on the applied pressure; at sea level water boils at 100 °C, at higher altitudes the temperature of the boiling point is lower.

The heat involved in a liquid-vapor phase change at the boiling point is determined by the formula:

$$\boldsymbol{Q = rm.} \tag{9.8}$$

where r is the ***heat of vaporization***. It is the heat required to change 1.0 kg of a substance from the liquid state to the vapor phase at the boiling point. For water it is 2260 kJ/kg.

9.6. THE HEAT BALANCE EQUATION

A closed system is an ***isolated system*** if no energy in any form passes across its boundaries. Because the total energy of the system cannot change, the heat lost by one part of the system is equal to the heat gained by the other part and sum of all these heat transfers is equal to zero:

$$\boldsymbol{Q_1 + Q_2 + \ldots + Q_n = 0.} \tag{9.9}$$

It is the heat balance equation for isolated system.

For not isolated system the heat balance equation is

$$\boldsymbol{Q_1 + Q_2 + \ldots + Q_n = Q_{ext},} \tag{9.10}$$

where $\boldsymbol{Q}_{ext}$ is an external heat transferred to system.

Example 9.2.

How much heat input is needed to raise the temperature of an 10-kg vat made of iron filled with 20 kg of water from 10 °C to 90 °C?

Solution. The specific heat of iron is c_1 = 450 J/kg·K°.

Thus: $Q_1 = c_1 m_1 \Delta T = 10 \text{ kg} \cdot 450 \dfrac{\text{j}}{\text{kg} \cdot \text{K}} (90 - 10) \text{ K} = 360 \text{ kJ}.$

The specific heat of water is c_w = 4200 J/kg·K°.

$$Q_w = c_w m_w \Delta T = 20 \text{ kg} \cdot 4200 \frac{\text{j}}{\text{kg} \cdot \text{K}} (90 - 10) \text{ K} = 6720 \text{ kJ}.$$

$$Q = Q_1 + Q_w = 360 \text{ kJ} + 6720 \text{ kJ} = 7080 \text{ kJ}.$$

Example 9.3.

A 0,5-kg chunk of ice at –10°C is placed in 3 kg of water at 20 °C. At what temperature and in what phase will the final mixture be? Ignore any heat flow to the surroundings, including the container.

Solution. First, check to see if the final state will be all ice, a mixture of ice and water at 0 °C, or all water.

To bring the 3 kg of water at 20 °C down to 0 °C would require an energy release of:

$$Q_w = m_w c_w (20\ °\text{C} - 0\ °\text{C}) = 3 \text{ kg} \cdot 4200 \text{ J/kg} \cdot \text{C°} \cdot 20\ °\text{C} = 252 \text{ kJ}.$$

On the other hand, to raise the ice temperature from –10 °C to 0 °C would require:

$Q_{ice} = m_{ice}c_{ice}(0\ °C – (–10\ °C)) = 0.5\ kg \cdot 2100\ J/kg \cdot C° \cdot 10\ °C = 10.5\ kJ.$

To change the ice to water at 0°C would require:

$$Q_F = \lambda m_{ice} = 0.5\ kg \cdot 333\,\frac{kJ}{kg} = 167\ kJ.$$

For a total: $Q_{ice} + Q_F = Q_w$;

$$10.5\ kJ + 167\ kJ = 177\ kJ.$$

This is not enough energy to cool the 3 kg of water from 20 °C to 0 °C, so that mixture will stay water, somewhere between 0 °C and 20 °C.

To determine the final temperature T:

$$m_{ice}c_{ice}(0\ °C – (–10\ °C)) + \lambda m_{ice} + m_{ice}c_w(T – 0\ °C) = m_w c_w(20\ °C – T)$$

or $$Q_{ice} + Q_F + T(m_{ice}c_{ice} + m_w c_w) = m_w c_w \cdot 20\ °C.$$

Solving for T we obtain: T = 5 °C.

9.7. THERMAL EXPANSION. COEFFICIENT OF LINEAR AND VOLUME EXPANSION

Most substances expand when heated and contract when cooled. However, the amount of expansion or contraction varies, depending on the material. The change in length $\boldsymbol{\Delta l}$ is directly proportional to the change in temperature $\boldsymbol{\Delta T}$ and the original length of the object $\boldsymbol{l_0}$:

$$\boldsymbol{\Delta l = \alpha l_0 \Delta T}, \qquad (9.11)$$

where **α** is the proportionally constant, which is called the ***coefficient of linear expansion*** for the particular material and has units of $\mathbf{K^{-1}}$.

We can write this proportionality in another form:

$$\boldsymbol{l = l_0(1 + \alpha\Delta T)}. \qquad (9.12),$$

The change in volume of a material that undergoes a temperature change is given by a similar relation:

$$\boldsymbol{\Delta V = \beta V_0 \Delta T}, \qquad (9.13),$$

where $\boldsymbol{\Delta T}$ is the change in temperature, $\boldsymbol{V_0}$ is the original volume and **β** is the coefficient of volume expansion. The units of **β** are K^{-1}.

Example 9.4.

The steel bed of a suspension bridge is 200 m long at 20 °C. If the extremes of temperature to which it might be exposed are –30 °C to +40 °C, how much will it contract and expand?

Solution. For steel $\alpha = 12 \cdot 10^{-6}\ (K)^{-1}$.

When the temperature increases to 40 °C:

$\Delta l_1 = \alpha l_0 \Delta T = 12 \cdot 10^{-6}\ (K)^{-1} \cdot 200\ m \cdot (40\ °C – 20\ °C) = 4.6 \cdot 10^{-2}\ m.$

When the temperature decreases to –30 °C:

$\Delta l_2 = \alpha l_0 \Delta T = 12 \cdot 10^{-6}\ (K)^{-1} \cdot 200\ m \cdot (–30\ °C – 20\ °C) = –12 \cdot 10^{-2}\ m.$

The total range the expansion is: $\Delta l = 4.6 \cdot 10^{-2}\ m + 12 \cdot 10^{-2}\ m \approx 17\ cm.$

TESTS

1. The first law of thermodynamics is concerned with the conservation of:

a) number of molecules; b) energy;
c) number of moles; d) temperature.

2. The addition of heat to a system appears as:

a) only increase in internal energy;
b) partly increase in internal energy and partly work done by the system;
c) only work done by the system;
d) all of these;
e) none of these.

3. In the equation for first law of thermodynamic $\Delta U = Q + W$, the ΔU represents:

a) change in internal energy; b) change in external energy;
c) both of them; d) none of these.

4. The internal energy of a ideal gas does not change during:

a) isothermal process; b) isobaric process;
c) isochoric process; d) none of these.

PROBLEMS

1. A steel railroad track has a length of 30 m when the temperature is 0 °C. What is its length when the temperature is 40 °C? The coefficient of linear expansion for steel is equal to $12 \cdot 10^{-6}$ K^{-1}. (Answer: 30,013 m)

2. An aluminum sphere is 8.75 cm in diameter. What will be its change in volume if it is heated from 30 °C to 180 °C? The coefficient of volume expansion for aluminum is equal to $75 \cdot 10^{-6}$ K^{-1}. (Answer: 3,9 cm^3)

3. How much external work can be done by a gas when it expands from 0.003 m^3 to 0.04 m^3 in volume under a constant pressure of 400 kPa? (Answer: 14800 J)

4. An engineer wishes to determine the specific heat of a new metal alloy. A 0,15-kg sample of the alloy is heated to 540 °C. It is then quickly placed in 0.4 kg of water at 10 °C, which is contained in a 0.2-kg aluminum calorimeter cup. The final temperature of the system is 30.5 °C. Calculate the specific heat of the alloy. (Answer: 497 J/kg·°C)

5. To what temperature will 8700 J of heat raise 3 kg of water that is initially at 10 °C? (Answer: 10,7°C)

6. When a 290-g piece of iron at 180 °C is placed in a 95-g aluminum calorimeter cup containing 250 g of glycerin at 10 °C, the final temperature is observed to be 38 °C. Estimate the specific heat of glycerin. (Answer: $2{,}3 \cdot 10^3$ J/kg·°C)

7. How much heat must be absorbed by ice of mass m = 720 g at –10 °C to take it to the liquid state at 15 °C? (Answer: 300 kJ)

10. ELECTRICITY

10.1. ELECTRIC CHARGE

There are two types of observed electric charge, which are designated as positive and negative. The symbol for charge is "***q***". The SI unit of charge is ***coulomb*** (C): 1 C = 1 A·1 sec.

The smallest charge found in nature is the charge of proton, it is given the symbol "***e*** " and is often referred to as the ***elementary charge***: $\boldsymbol{e = 1{,}6 \cdot 10^{-19}\ C}$. The charge on the electron is "–***e***". Electric charge is thus said to be quantized (existing only in discrete amounts: *1e*, *2e*, *3e*, etc.). Any net charge ***q*** (negative or positive) can be determined as: $\boldsymbol{q = n \cdot e}$, and $\boldsymbol{n = 1, 2, 3}$…

10.2. LAW OF CONSERVATION OF ELECTRIC CHARGE

The law of conservation of electric charge makes sense for the isolated system that does not interact with or receive charge from other systems. In an isolated system, the total electric charge of the system is equal to the algebraic sum of all electric charges $\boldsymbol{q_1, q_2, \ldots q_i}$ located in the system:

$$\boldsymbol{q_1, q_2, \ldots q_i = \mathrm{const.}}$$

The law states that the total electric charge in an isolated system always remains constant, regardless of other possible changes within the system.

10.3. COULOMB'S LAW

When the electric charges have likely signs there is a repulsive force between them and, opposite, when the charges are unlikely, there is attractive force between them. The force between two charged small spheres was studied by Coulomb. Coulomb's Law states that the electrostatic force ***F*** in vacuum between two point charges $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ is directly proportional to the product of the magnitude of the charges and inversely proportional to the square of the distance ***r*** between them:

$$\boldsymbol{F = k\frac{q_1 q_2}{r^2}}, \tag{10.1}$$

where $\boldsymbol{k = 9 \cdot 10^9\, \frac{N \cdot m^2}{C^2}}$ is the electrostatic constant.

The direction of forces is always along the line joining the two point charges, and it is attractive if the charges are opposite and repulsive if the charges are like.

This formula (10.1) only applies to point charges (spatial size negligible compared to other distances) or spherically charges when they are at rest.

The electrostatic constant $\boldsymbol{k}$ in equation (10.1) is often written in terms of another constant, $\boldsymbol{\varepsilon_0}$, called the ***permittivity of free space*** (i. e. vacuum). It is related to $\boldsymbol{k}$ by: $\boldsymbol{k = 1/4\pi\varepsilon_0}$. So in air or vacuum Coulomb's law can then be written as

$$\boldsymbol{F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}},$$

where

$$\boldsymbol{\varepsilon_0 = \frac{1}{4\pi k} = 8.85\cdot 10^{-12}\frac{C^2}{N\cdot m^2}}. \qquad (10.2)$$

If the charges are situated in a medium of permittivity relative $\boldsymbol{\varepsilon}$, then the magnitude of the interaction force $\boldsymbol{F_m}$ between them will be less:

$$\boldsymbol{F_m = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\varepsilon r^2}}. \qquad (10.3)$$

Dividing equation (10.2) by (10.3) one can obtain:

$$\boldsymbol{\frac{F}{F_m} = \varepsilon > 1} \quad \text{or} \quad \boldsymbol{F_m = \frac{F}{\varepsilon}}. \qquad (10.4)$$

The value $\boldsymbol{\varepsilon_a = \varepsilon\cdot\varepsilon_0}$ is the ***absolute permittivity of the medium***.
Coulomb's law describes the ***electrostatics*** force between two charges at rest.

***Example 10.1*. Electric force on electron by proton.**

Determine the magnitude of the electric force on the electron of a hydrogen atom exerted by the single proton ($\boldsymbol{q_2} = +e$) that is its nucleus. Assume the electron "orbits" the proton at its average distance of $\boldsymbol{r} = 0.53\cdot 10^{-10}$ m.

Solution. We use Coulomb's law $F = k\dfrac{q_1 q_2}{r^2}$ with $\boldsymbol{r} = 0.53\cdot 10^{-10}$ m and $\boldsymbol{q_1} = \boldsymbol{q_2} = 1.6\cdot 10^{-19}$ C (ignoring the signs of the charges):

$$F = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(1.6\cdot 10^{-19}\,\mathrm{C})(1.6\cdot 10^{-19}\,\mathrm{C})}{(0.53\cdot 10^{-10}\,\mathrm{m})^2} = 8.2\cdot 10^{-8}\,\mathrm{N}.$$

The direction of the force on the electron is toward the proton, since the charges have opposite signs and the force is attractive.

***Example 10.2*. Which charge exerts the greater force?**

Two positive point charges, $\boldsymbol{q_1}$ = 50 μC and $\boldsymbol{q_2}$ = 1 μC, are separated by a distance $\boldsymbol{l}$. Which is larger in magnitude, the force that $\boldsymbol{q_1}$ exerts on $\boldsymbol{q_2}$, or the force that $\boldsymbol{q_2}$ exerts on $\boldsymbol{q_1}$?

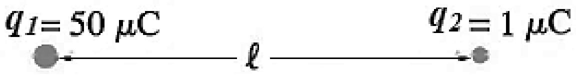


Fig. 10.1. Example 10.2

Solution. From Coulomb's law, the force on $\boldsymbol{q_1}$ exerted by $\boldsymbol{q_2}$ is $F_{12} = k\dfrac{q_1 q_2}{l^2}$.

The force on q_2 exerted by q_1 is $F_{21} = k\dfrac{q_2 q_1}{l^2}$, which is the same magnitude. The equation is symmetric with respect to the two charges, so $F_{21} = F_{12}$.

NOTE. *Newton's third law also tells us that these two forces must have equal magnitude.*

10.4. THE ELECTRIC FIELD. THE ELECTRIC FIELD STRENGTH

Electric field is said to exist in the region of space around a charged object: the source of electric field is a charge. The presence of an electric field may be detected with another charge. When a positive test charge q_0 is placed near a charge q, which is the source of electric field, an electrostatic force F will act on the test charge (fig. 10.2) and $F \sim q_0$.

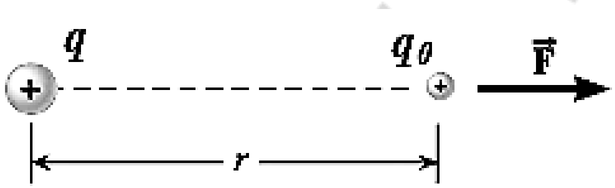


Fig. 10.2. A test charge q_0 is placed near a charge q, which is the source of electric field, an electrostatic force F acts on the test charge

But the ratio
$$\vec{E} = \frac{\vec{F}}{q_0} \qquad (10.5)$$
does not depend on the test charge q_0 and is called the ***electric field strength*** or ***electric field intensity*** (or electric field).

The **electric field strength** $\vec{E}$ is a **vector** whose direction is the direction of the force acting on a positive test charge q_0 placed at the point, and whose magnitude is the ***force per unit charge***.

Thus E has SI units of Newtons per Coulomb (N/C). If q_0 is positive, F and E will point in the same direction. If q_0 is negative, F and E point in opposite direction (fig. 10.3).

Fig. 10.3. If q_0 is positive, F and E will point in the same direction. If q_0 is negative, F and E point in opposite direction

The electric field strength $\boldsymbol{E}$ at a distance $\boldsymbol{r}$ from single point charge $\boldsymbol{q}$ can be written as:

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{\boldsymbol{q_0}} = \boldsymbol{k}\frac{\boldsymbol{q_0 q}}{\boldsymbol{q_0\ \varepsilon r^2}} = \boldsymbol{k}\frac{\boldsymbol{q}}{\boldsymbol{\varepsilon \cdot r^2}}. \qquad (10.6)$$

If the electric field strength $\boldsymbol{E}$ is due to more than one charge ($\boldsymbol{q_1, q_2, q_3, \ldots q_n}$), the individual electric field strengths (call them $\boldsymbol{E_1, E_2, \ldots E_n}$) due to each charge are added as vectors to get the total electric field strength $\boldsymbol{E}$ at any point of field (fig. 10.4). It is **the principle of superposition**:

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2 + \vec{\boldsymbol{E}}_3 + \ldots + \vec{\boldsymbol{E}}_n. \qquad (10.7)$$

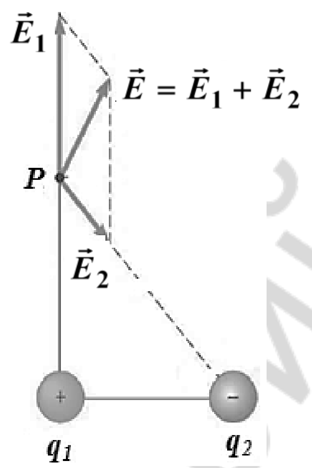


Fig. 10.4. At any point $\boldsymbol{P}$, the total electric field strength due to the charges $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ equals the vector sum of electric field strengths of the charges: $\boldsymbol{E = E_1 + E_2}$. The direction of the individual electric field strengths is the direction of the force on a positive test charge

***Example 10.3. E* at a point between two charges.**

Two point charges are separated by a distance of 10.0 cm. One has a charge of –25 μC and the other +50 μC. (a) Determine the direction and magnitude of the electric field at a point $\boldsymbol{P}$ between the two charges that is 2.0 cm from the negative charge (fig. 10.5, *a*). (*b*) If an electron (mass = $9.11 \cdot 10^{-31}$ kg) is placed at rest at $\boldsymbol{P}$ and then released, what will be its initial acceleration (direction and magnitude)?

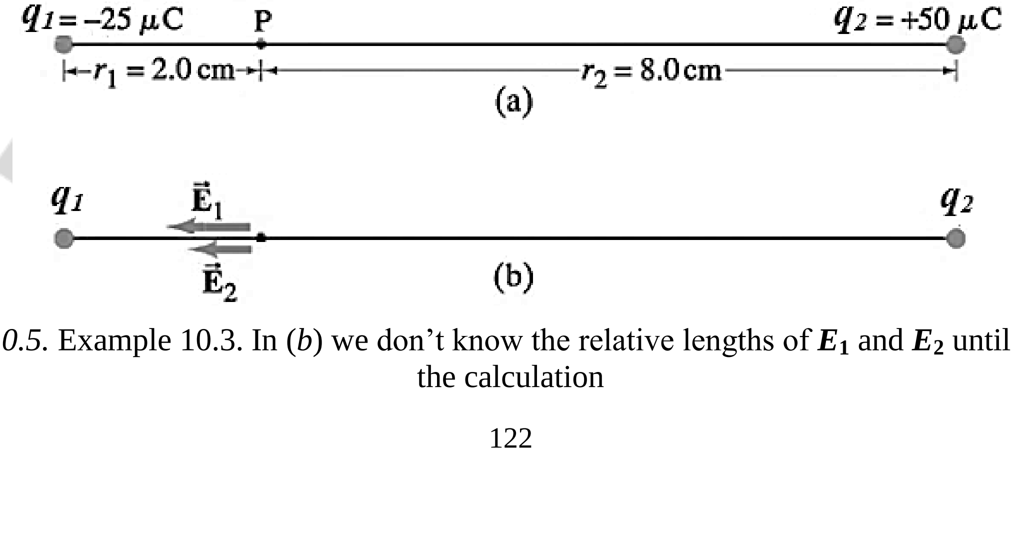


Fig. 10.5. Example 10.3. In (*b*) we don't know the relative lengths of $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ until we do the calculation

Solution. The electric field at ***P*** will be the vector sum of the fields created separately by $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$. The field due to the negative charge $\boldsymbol{q_1}$ points toward $\boldsymbol{q_1}$, and the field due to the positive charge $\boldsymbol{q_2}$ points away from $\boldsymbol{q_2}$. Thus both fields point to the left as shown in Figure b and we can add the magnitudes of the two fields together algebraically, ignoring the signs of the charges. In fig. 10.5, *b* we use Newton's second law ($F = ma$) to determine the acceleration, where $F = qE$.

a) Each field is due to a point charge as given by $E = kq/r^2$. The total field is

$$E = k\frac{q_1}{r_1^2} + k\frac{q_2}{r_2^2} = k\left(\frac{q_1}{r_1^2} + \frac{q_2}{r_2^2}\right) = (9.0 \cdot 10^9\,\mathrm{N \cdot m^2/C^2})\left(\frac{25 \cdot 10^{-6}\,\mathrm{C}}{(2.0 \cdot 10^{-2}\,\mathrm{m})^2} + \frac{50 \cdot 10^{-6}\,\mathrm{C}}{(8.0 \cdot 10^{-2}\,\mathrm{m})^2}\right) =$$

$$= 6.3 \cdot 10^8\,\mathrm{N/C}.$$

b) The electric field points to the left, so the electron will feel a force to the *right* since it is negatively charged. Therefore the acceleration $a = F/m$ (Newton's second law) will be to the right. The force on a charge $\boldsymbol{q}$ in an electric field $\boldsymbol{E}$ is $F = qE$. Hence the magnitude of the acceleration is

$$a = \frac{F}{m} = \frac{qE}{m} = \frac{(1.60 \cdot 10^{-19}\,\mathrm{C})(6.3 \cdot 10^8\,\mathrm{N/C})}{9.11 \cdot 10^{-31}\,\mathrm{kg}} = 1.1 \cdot 10^{20}\,\mathrm{m/s^2}.$$

NOTE. *By carefully considering the directions of each field (**E_1** and **E_2**) before doing any calculations, we made sure our calculation could be done simply and correctly.*

Example 10.4. E **above two point charges.**

Calculate the total electric field (a) at point ***A*** and (b) at point ***B*** in fig. 10.6 due to both charges, $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$.

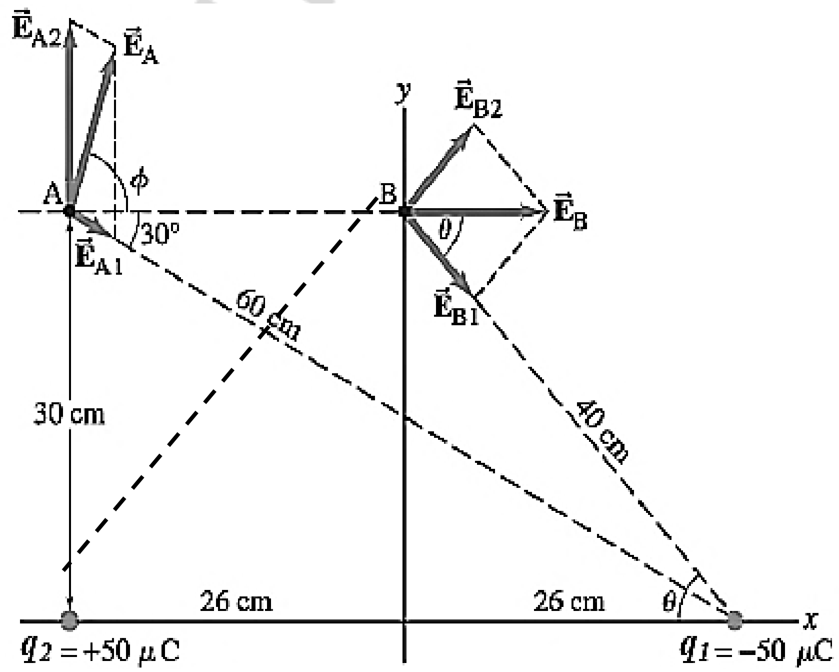


Fig. 10.6. Calculation of the electric field at points ***A*** and ***B*** for Example 10.4

Solution. The calculation is much like that of Example 9.5, except now we are dealing with electric fields instead of force. The electric field at point ***A*** is the vector sum of the fields $\boldsymbol{E_{A1}}$ due to $\boldsymbol{q_1}$, and $\boldsymbol{E_{A2}}$ due to $\boldsymbol{q_2}$. We find the magnitude of the field produced by each point charge, then we add their components to find the total field at point ***A***. We do the same for point ***B***.

a) The magnitude of the electric field produced at point ***A*** by each of the charges $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ is given by $E = kq/r^2$, so

$$E_{A1} = k\frac{q}{r^2} = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(50\cdot 10^{-6}\,\mathrm{C})}{(0.60\ \mathrm{m})^2} = 1.25\cdot 10^6\,\mathrm{N/C},$$

$$E_{A2} = k\frac{q}{r^2} = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(50\cdot 10^{-6}\,\mathrm{C})}{(0.30\ \mathrm{m})^2} = 5.0\cdot 10^6\,\mathrm{N/C}.$$

from ***A*** away from $\boldsymbol{q_2}$, as shown; so the total electric field at ***A***, $\boldsymbol{E_A}$ has components

$$E_{Ax} = E_{A1}\cos 30^0 = 1.1\cdot 10^6\,\mathrm{N/C},$$

$$E_{Ay} = E_{A2}\sin 30^0 = 4.4\cdot 10^6\,\mathrm{N/C}.$$

Thus the magnitude of $\boldsymbol{E_A}$ is

$$E_A = \sqrt{(1.1)^2 + (4.4)^2}\cdot 10^6\,\mathrm{N/C} = 4.5\cdot 10^6\,\mathrm{N/C},$$

and its direction is ϕ given by $\tan\phi = E_{Ay}/E_{Ax} = 4.4/1.1 = 4.0$, so $\phi = 76°$.

b) Because ***B*** is equidistant from the two equal charges (40 cm by the Pythagorean theorem), the magnitudes of $\boldsymbol{E_{B1}}$ and $\boldsymbol{E_{B2}}$ are the same; that is,

$$E_{B1} = E_{B2} = k\frac{q}{r^2} = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(50\cdot 10^{-6}\,\mathrm{C})}{(0.40\ \mathrm{m})^2} = 2.8\cdot 10^6\,\mathrm{N/C}.$$

Also, because of the symmetry, the ***y*** components are equal and opposite, and so cancel out. Hence the total field $\boldsymbol{E_B}$ is horizontal and equals

$$E_{B1}\cos\theta + E_{B2}\cos\theta = 2E_{B1}\cos\theta.$$

From the diagram, $\cos\theta = 26\ \mathrm{cm}/40\ \mathrm{cm} = 0.65$. Then

$$E_B = 2E_{B1}\cos\theta = 2(2.8\cdot 10^6\,\mathrm{N/C})(0.65) = 3.6\cdot 10^6\,\mathrm{N/C},$$

and the direction of $\boldsymbol{E_B}$ is along the ***+x*** direction.

NOTE. *We could have done part (b) in the same way we did part (a). But symmetry allowed us to solve the problem with less effort.*

Since the electric field strength is a vector, it is sometimes referred to as vector field. Lines of electric field strength are a convenient way of visualizing the electric field. Lines of electric field strength indicate the direction of the force due to the given field on a positive test charge. For a positive point charge, the electric field strength lines are directed radially outward from the charge (fig. 10.7). For a negative point charge they point radially inward toward the charge because that is the direction the force would be on a positive test charge in each case. Since the electric field strength is the electric force per unit charge, the electric field strength lines are sometimes called ***lines of force***.

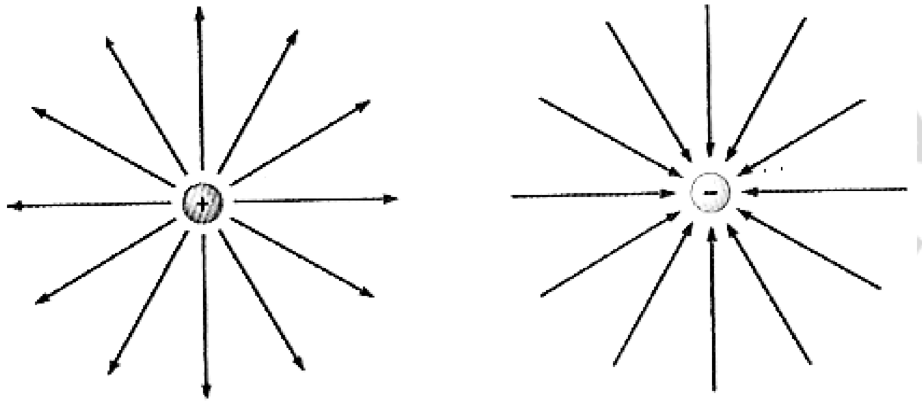


Fig. 10.7. The electric field strength lines near a single positive point charge and negative one. Electric field lines extend away from positive charge (where they originate) and toward negative charge (where they terminate)

The number of lines starting on a positive charge, or ending on a negative charge, is proportional to the magnitude of the charge. Notice that near the charge, where the electric field strength is greatest, the lines are closer together. This is a general property of electric field strength lines: the closer the lines are together, the stronger the electric field strength in that region.

The uniform electric field is electric field where the vector $\boldsymbol{E}$ is constant everywhere in magnitude and direction. Thus electric fields are drown with parallel, equally spaced electric field strength lines.

Example 10.5. **Electron accelerated by electric field.**

An electron (mass $m = 9.1 \cdot 10^{-31}$ kg) is accelerated in the uniform field $\boldsymbol{E}$ ($E = 2.0 \cdot 10^4$ N/C) between two parallel charged plates. The separation of the plates is 1.5 cm. The electron is accelerated from rest near the negative plate and passes through a tiny hole in the positive plate. (a) With what speed does it leave the hole? (b) Show that the gravitational force can be ignored. Assume the hole is so small that it does not affect the uniform field between the plates.

Solution. We can obtain the electron's velocity using the kinematic equations, after first finding its acceleration from Newton's second law, $F = ma$. The magnitude of the force on the electron is $F = qE$ and is directed to the right.

a) The magnitude of the electron's acceleration is

$$mg = (9.1 \cdot 10^{-31}\text{kg})(9.8\ \text{m/s}^2) = 8.9 \cdot 10^{-30}\,\text{N}.$$

Between the plates $\boldsymbol{E}$ is uniform so the electron undergoes uniformly accelerated motion with acceleration

$$a = \frac{(1.6 \cdot 10^{-19}\text{C})(2.0 \cdot 10^4\,\text{N/C})}{9.1 \cdot 10^{-31}\text{kg}} = 3.5 \cdot 10^{15}\text{m/s}^2.$$

It travels a distance x = 1.5·10^{-2} m before reaching the hole, and since its initial speed was zero, we can use the kinematic equation, $\upsilon^2 = \upsilon_0^2 + 2ax$, with $\upsilon_0 = 0$: $\upsilon = \sqrt{2ax} = \sqrt{2(3.5\cdot10^{15}\,\text{m/s}^2)(1.5\cdot10^{-2}\,\text{m})} = 1.0\cdot10^{7}\,\text{m/s}.$

There is no electric field outside the plates, so after passing through the hole, the electron moves with this speed, which is now constant.

b) The magnitude of the electric force on the electron is

$$qE = (1.6\cdot10^{-19}\,\text{C})(2.0\cdot10^{4}\,\text{N/C}) = 3.2\cdot10^{-15}\,\text{N}.$$

The gravitational force is $mg = (9.1\cdot10^{-31}\ \text{kg})(9.8\ \text{m/s}^2) = 8.9\cdot10^{-30}\,\text{N}$ which is 10^{14} times smaller! Note that the electric field due to the electron does not enter the problem (since a particle cannot exert a force on itself).

PROBLEMS

1. Calculate the magnitude of the force between two 2.50 C point charges 3.0 m apart in air. (Answer: 6.25·10^{+9} N)

2. The force between two charges in free space is 5 N. What will the force between them be if they are in a medium of relative permittivity 2? (Answer: 2.5 N)

3. Calculate the magnitude of the electric force between an iron nucleus (q = +26e) and its innermost electron if the distance between them is 1.5·10^{-12} m. (Answer: 2.66·10^{-3} N)

4. Particles of charge +88, –55, and +70 μC are placed in a line fig. 10.8. The center one is 0.75 m from each of the others. What are the net force on each charge due to the other two? (Answer: 52.8 N, 15.84 N, 36.9 N)

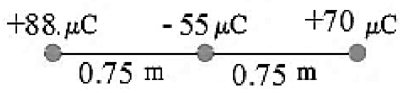


Fig. 10.8

5. Find the magnitude and direction of the electric field strength at points A and B in fig. 10.9 due to the two positive charges (q = 7 μC). (Answer: 4.5·10^{+6} N/C)

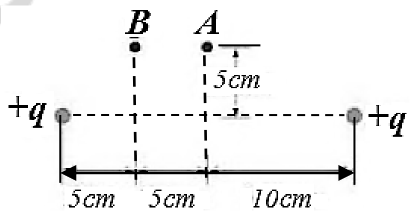


Fig. 10.9

6. How many electrons make up a charge of 100 μC? (Answer: 6.25·10^{+14})

7. Four equal point charges of +3 μC are placed at the four corners of a square that is 40 cm on a side. Find the force on an one of the charges? (Answer: 0.97 N)

9. Two charges of +1μC and –1μC are placed at the corners of the base of an equilateral triangle. The length of a side of the triangle is 0.7 m. Find the electric field strength at the apex of the triangle. (Answer: 18.4 kN/C)

8. What must the charge (sign and magnitude) of a particle of mass 5g be for it to remain stationary when placed in a downward-directed electric field of magnitude 800 N/C? (Answer: –0.0000613 C)

TESTS

1. The conservation of electric charge implies that:
a) charge can't be created;
b) charge can't be destroyed;
c) the number of charged particles in the universe is constant;
d) simultaneous creation of equal and opposite charges is permissible.

2. Two charges are placed at a certain distance apart. If a dielectric slab is placed between them, what happens to the force between the charges?
a) decreases;
b) increases;
c) remains unchanged;
d) may increase or decrease depending on the nature of the dielectric.

3. Coulomb's law is given by $F = kq_1q_2r^n$, where n is:
a) 1/2; b) –2; c) 2; d) $-^1/_2$.

10.5. ELECTRIC POTENTIAL AND POTENTIAL DIFFERENCE

Any charged body has an electric potential energy $\boldsymbol{W_{pot}}$ in electric field, which is directly proportional to the magnitude of the charge $\boldsymbol{q_0}$: $\boldsymbol{W_{pot} \sim q_0}$. But the ratio $\boldsymbol{W_{pot}/q_0}$ doesn't depend on the charge magnitude placed in the electric field. The ratio $\boldsymbol{W_{pot}/q_0}$ is the energy characteristic of the electric field and is called electric potential φ:

$$\varphi = \frac{W_{pot}}{q_0}. \quad (10.8)$$

Electric potential **φ** is the potential energy per unit charge at a point in an electric field. Electric potential **φ** is a scalar characteristic of an electric field. Unit of electric potential is ***Volt (V)*** (1 Volt = 1 Joule per Coulomb (J/C)). The electric potential **φ** at a distance ***r*** from a single point charge is

$$\varphi = k\frac{q}{\varepsilon \cdot r} = \frac{q}{4\pi\varepsilon\varepsilon_0 r}. \quad (10.9)$$

The potential from a collection of ***n*** charges is the algebraic sum of the potential due to each charge separately (this is much easier to calculate than the net electric field strength, which would be a vector sum). Potential due to a group of point charges $\boldsymbol{q_1, q_2, q_3, \ldots q_n}$ can be found as:

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \ldots + \varphi_n. \quad (10.10)$$

***Example 10.6*. Potential due to the two charges.**

Calculate the electric potential (a) at point ***A*** in fig. 10.10 due to the two charges shown, and (b) at point ***B***.

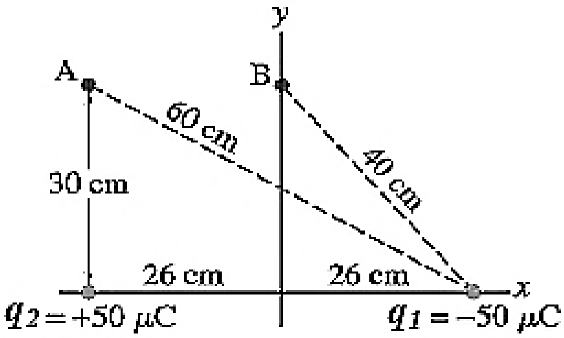


Fig. 10.10. Example 10.6

Solution. The total potential at point ***A*** (or at point ***B***) is the sum of the potentials at that point due to each of the two charges $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$. The potential due to each single charge is given by $\varphi = k\dfrac{q}{r}$.

We do not have to worry about directions because electric potential is a scalar quantity. But we do have to keep track of the signs of charges.

a) We add the potentials at point ***A*** due to each charge $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$:

$$\varphi_A = \varphi_{A2} + \varphi_{A1} = k\frac{q_2}{r_{2A}} + k\frac{q_1}{r_{1A}}.$$

where $\boldsymbol{r_{1A}}$ = 60 cm and $\boldsymbol{r_{2A}}$ = 30 cm. Then

$$\varphi_A = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(5.0\cdot 10^{-5}\,\mathrm{C})}{0.30\ \mathrm{m}} + \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C})(-5.0\cdot 10^{-5}\,\mathrm{C})}{0.60\ \mathrm{m}} =$$

$$= 1.50\cdot 10^6 V - 0.75\cdot 10^6 V = 7.5\cdot 10^5 V.$$

b) At point ***B***, $\boldsymbol{r_{1B}} = \boldsymbol{r_{2B}}$ = 0.40 m, so

$$\varphi_B = \varphi_{B2} + \varphi_{B1} = \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C^2})(5.0\cdot 10^{-5}\,\mathrm{C})}{0.40\ \mathrm{m}} +$$

$$+ \frac{(9.0\cdot 10^9\,\mathrm{N\cdot m^2/C})(-5.0\cdot 10^{-5}\,\mathrm{C})}{0.40\ \mathrm{m}} = 0V.$$

NOTE. *The two terms in the sum in (b) cancel for any point equidistant from* $\boldsymbol{q_1}$ *and* $\boldsymbol{q_2}$ *(*$\boldsymbol{r_{1B}} = \boldsymbol{r_{2B}}$*). Thus the potential will be zero everywhere on the plane equidistant between the two opposite charges. This plane where* $\boldsymbol{\varphi}$ *is constant is called an equipotential surface.*

If all the points of a surface are at the same electric potential, then the surface is called an equipotential surface. In case of an isolated point charge, all points equidistant from the charge are at same potential. Thus, equipotential surfaces in this case will be a series of concentric spheres with the point charge as their centre (fig. 10.11). The potential will however be different for different spheres.

Work ***A*** done by electric field in moving a unit positive charge is equal to potential energy difference:

$$A = W_{\mathrm{pot1}} - W_{\mathrm{pot2}} = q_0\,(\varphi_1 - \varphi_2) = q_0 U. \qquad (10.11)$$

The potential difference $\boldsymbol{U}$ between point ***1*** and point ***2*** is:

$$U = \varphi_1 - \varphi_2 = \frac{A}{q_0}. \quad (10.12)$$

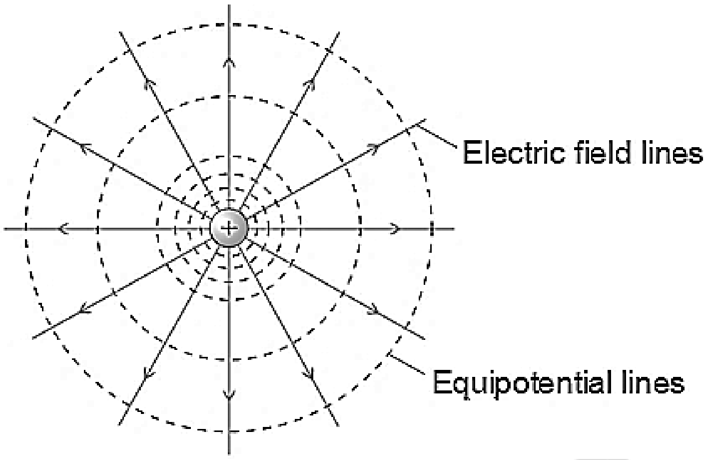


Fig. 10.11. The equipotential lines and electric field lines

The potential difference $\boldsymbol{U}$ between two points in an electric field is defined as the amount of work $\boldsymbol{A}$ done in moving a unit positive charge $\boldsymbol{q_0}$ from one point to the other. The unit of potential difference $\boldsymbol{U}$ is ***Volt*** (1 V = 1 J/1 C). The electric potential **φ** of an electric field at infinity from charge is equal to zero.

Suppose a positive test charge $\boldsymbol{q_0}$ is moved from point ***1*** to point ***2*** in a uniform electric field between the two charged plates (fig. 10.12).

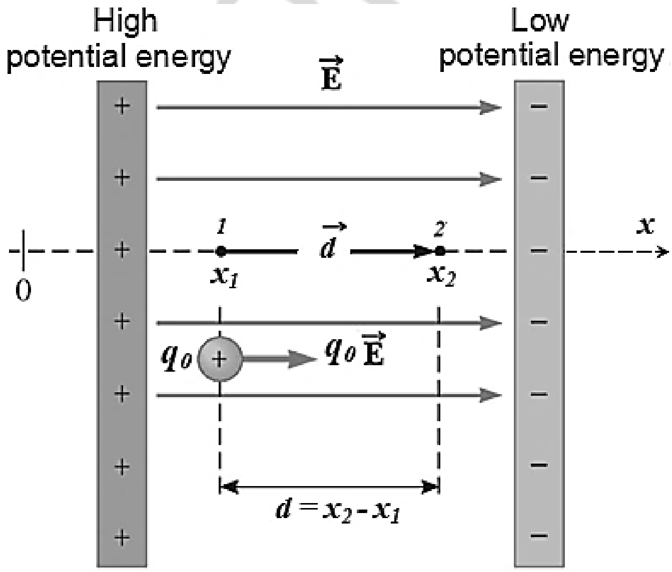


Fig. 10.12. The charge $\boldsymbol{q_0}$ is moved from point ***1*** to point ***2*** in a uniform electric field between the two charged plates

The work $\boldsymbol{A}$ done by an electric force $\boldsymbol{F}$ or "field" in moving a positive test charge $\boldsymbol{+q_0}$ along the electric field line at a distance $\boldsymbol{d = x_2 - x_1}$ is:

$$A = F \cdot d = q_0 \cdot E \cdot d. \quad (10.13)$$

Thus, in uniform electric field the relation between potential difference ***U*** and electric field strength ***E*** is the following:

$$U = E \cdot d, \tag{10.14}$$

where ***d*** is the distance, parallel to the field lines, between two points.

The equation (10.14) shows that the unit for electric field strength ***E*** is volt per meter. Thus, the following relation among units is valid: 1 N/C = 1 V/m.

The electric force is found to be a conservative force. Therefore, the work ***A*** done by the electrostatic force does not depend upon the path chosen to move charge from point ***1*** to point ***2***, but it is determined by the potential difference ***U*** between point ***1*** and point ***2***.

***Example 10.7*. Work required to bring two positive charges close together.**

What minimum work must be done by an external force to bring a charge $\boldsymbol{q_0}$ = 3.00 μC from a great distance away (take $\boldsymbol{r} = \infty$) to a point 0.500 m from a charge $\boldsymbol{q}$ = 20.0 μC?

Solution. To find the work we can not simply multiply the force times distance because the force is not constant. Instead we can set the change in potential energy equal to the work required of an *external* force, and equation $A = -(W_{pot1} - W_{pot2}) = -q_0(\varphi_b - \varphi_a)$. We get the potentials $\boldsymbol{\varphi_b}$ and $\boldsymbol{\varphi_a}$ using $\varphi = k\dfrac{q}{r}$.

The work done by the electric field is equal to the change in potential energy:

$$A = -q_0(\varphi_b - \varphi_a) = -q_0\left(\frac{kq}{r_b} - \frac{kq}{r_a}\right),$$

where r_b = 0.500 m and $r_a = \infty$. The right-hand term within the parentheses is zero ($1/\infty = 0$) so

$$A = -(3.00 \cdot 10^{-6}\,\text{C})\frac{(8.99 \cdot 10^{9}\,\text{N} \cdot \text{m}^2/\text{C}^2)(2.00 \cdot 10^{-5}\,\text{C})}{0.500\ \text{m}} = -1.08\ \text{J}.$$

NOTE. *The electric field does negative work in this case. In order to bring the charge to this point, the external force would have to do work **A**= +1.08 J, assuming no acceleration of the charge.*

***Example 10.8*. Electron in TV tube.**

Suppose an electron in a cathode ray tube is accelerated from rest through a potential difference $\boldsymbol{\varphi_b - \varphi_a = \varphi_{ba}}$ = +5000 V (fig. 10.13). (a) What is the change in electric potential energy of the electron? (b) What is the speed of the electron ($m = 9.1 \cdot 10^{-31}$ kg) as a result of this acceleration? (c) Repeat for a proton ($m = 1.67 \cdot 10^{-27}$ kg) that accelerates through a potential difference φ_{ba} = –5000 V.

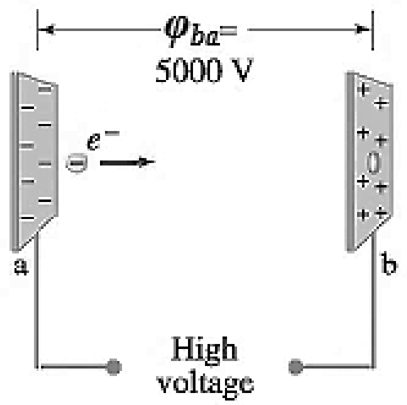


Fig. 10.13. Example 10.8. Electron accelerated in TV tube

Solution. The electron, accelerated toward the positive plate, will decrease in potential energy by an amount $\Delta W = q\varphi_{ba}$. The loss in potential energy will equal its gain in kinetic energy (energy conservation).

a) The charge on an electron is $\boldsymbol{q} = -e = -1.6 \cdot 10^{-19}$ C. Therefore its change in potential energy is

$\Delta W = q\varphi_{ba} = (-1.6 \cdot 10^{-19}\ \text{C}) \cdot (+5000\ \text{V}) = -8.0 \cdot 10^{-16}$ J.

The minus sign indicates that the potential energy decreases. The potential difference $\boldsymbol{\varphi_{ba}}$ has a positive sign since the final potential $\boldsymbol{\varphi_b}$ is higher than the initial potential $\boldsymbol{\varphi_a}$. Negative electrons are attracted toward a positive electrode and repelled away from a negative electrode.

b) The potential energy lost by the electron becomes kinetic energy ***K***. From conservation of energy, $\Delta K + \Delta W = 0$, so $\Delta K = -\Delta W$:

$$\frac{1}{2}m\upsilon^2 - 0 = -q(\varphi_b - \varphi_a) = -q\varphi_{ba},$$

where the initial kinetic energy is zero since we are given that the electron started from rest. We solve for $\boldsymbol{\upsilon}$:

$$\upsilon = \sqrt{-\frac{2q\varphi_{ba}}{m}} = \sqrt{-\frac{2(-1.6 \cdot 10^{-19}\text{C})(5000\ \text{V})}{9.1 \cdot 10^{-31}\text{kg}}} = 4.2 \cdot 10^7\ \text{m/s}.$$

c) The proton has the same magnitude of charge as electron, through of opposite sign. Hence for the same magnitude of $\boldsymbol{\varphi_{ba}}$ we expect the same change in ***W***, but a lesser speed since the proton's mass is greater. Thus:

$\Delta W = q\varphi_{ba} = (+1.6 \cdot 10^{-19}\ \text{C})(-5000\ \text{V}) = -8.0 \cdot 10^{-16}$ J

and
$$\upsilon = \sqrt{-\frac{2q\varphi_{ba}}{m}} = \sqrt{-\frac{2(1.6 \cdot 10^{-19}\text{C})(-5000\ \text{V})}{1.67 \cdot 10^{-27}\ \text{kg}}} = 9.8 \cdot 10^5\ \text{m/s}.$$

NOTE. *The electric potential energy does not depend on the mass, only on the charge and voltage. The speed does depend on* ***m****.*

***Example 10.9*. Uniform electric field obtained from voltage.**

Two parallel plates are charged to produce a potential difference of 50 V. If the separation between the plates is 0.050 m, calculate the magnitude of the electric field in the space between the plates.

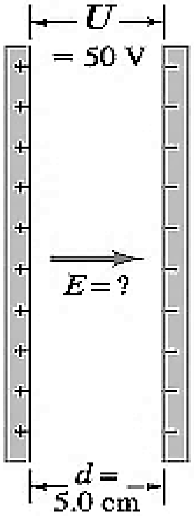


Fig. 10.14. Example 10.9

Solution. We apply equation $\boldsymbol{U = E \cdot d}$ to obtain the magnitude of $\boldsymbol{E}$, assumed uniform. The electric field magnitude is $E = U/d$ = (50 V/0.050 m) = 1000 V/m.

Problems

1. A 5.0g object carries a net charge of 3.8 μC. It acquires a speed υ when accelerated from rest through a potential difference $\boldsymbol{U}$. A 2.0 g object acquires twice the speed under the same circumstances. What is its charge? (Answer: 6.08 μC)

2. A particle with a charge of +8 nC is in a uniform electric field $\boldsymbol{E}$ directed to the left. It is released from rest and moves to the left. After it has moved 3 cm, its kinetic energy is found to be $5 \cdot 10^{-6}$ J. (a) What work was done by the electric force? (b) What is the potential of the starting point with respect to the endpoint? (c) What is the magnitude of $\boldsymbol{E}$? (Answer: a) $5 \cdot 10^{-6}$ J; b) 625 V; c) 20833 N/C)

3. The potential energy of an electron ($q = -1.6 \cdot 10^{-19}$ C) increases by $3.3 \cdot 10^{-15}$ J when it moves 3.5cm parallel to a uniform electric field. What is the magnitude of the electric field through which the electron passes? (Answer: $5.9 \cdot 10^{5}$ N/C)

Tests

1. The work needed to move a –7.0 μC charge from ground to a point whose potential is +6.00 V higher is:

a) $-4.2 \cdot 10^{-5}$ J; b) $+4.2 \cdot 10^{-5}$ J; c) 10^{-4} J; d) $-1.2 \cdot 10^{-5}$ J.

2. State which of the following is correct:

a) Joule = Coulomb · Volt; b) Joule = Coulomb / Volt;
c) Joule = Volt / Ampere; d) Joule = Volt · Ampere.

3. How much kinetic energy will an electron gain (in joules) if it falls through a potential difference of 21 000 V in a TV picture tube.

a) $3\cdot10^{+15}$ J; b) $3.4\cdot10^{-15}$ J; c) $2\cdot10^{-13}$ J; d) 3.4 J.

4. An electric field of 640 V/m is desired between two parallel plates 11.0 mm apart. A voltage which should be applied is:

a) 700 V; b) 0.7 V; c) 7.04 V; d) 10 V.

5. The potential difference which is needed to give a helium nucleus ($q = 3.2\cdot10^{-19}$ C) 48 keV of kinetic energy is:

a) $2.4\cdot10^{4}$ V; b) $4.8\cdot10^{3}$ V; c) 24 kV; d) 10^{3} V.

6. What is the potential energy of two equal positive charges of 1 μC each held 1 m apart in air?

a) $9\cdot10^{-3}$ J; b) $9\cdot10^{-3}$ C/V; c) 1 J; d) zero.

7. How strong is the electric field between two parallel plates 5.0 mm apart if the potential difference between them is 110 V?

a) $220\cdot10^{2}$ V/m; b) 110 V/m; c) $22\cdot10^{-3}$ V/m; d) $22\cdot10^{3}$ V/m.

8. An electron of mass ***m*** and charge ***e*** travels from rest through a potential difference of ***U*** volts. The final velocity of the electron is:

a) $\dfrac{2eU}{m}$; b) $\dfrac{2mU}{e}$; c) $\sqrt{\dfrac{2eU}{m}}$; d) $\sqrt{\dfrac{2mU}{e}}$.

10.6. CAPACITORS. CAPACITANCE. ELECTRIC ENERGY STORAGE

A capacitor is a device used to store electric charge, and usually consists of two conducting plates placed near each other but not touching. The parallel plate capacitor shown in fig. 10.15 has two identical conducting plates with a surface area ***S***, separated by a distance ***d*** (with dielectric material between the plates, for example air). The capacitor is symbolized by a set of parallel lines.

If a voltage is applied to a capacitor by connecting it to a battery as in fig. 10.15, the capacitor becomes charged quickly: one plate acquires a negative charge, the other an equal amount of positive charge. Each battery terminal and the plate of the capacitor connected to it are at the same potential: hence the full battery voltage appears across the capacitor. It is found that the amount of charge ***q*** acquired by each plate is proportional to the magnitude of the potential difference ***U*** between them:

$$\boldsymbol{Q = CU.} \qquad (10.15)$$

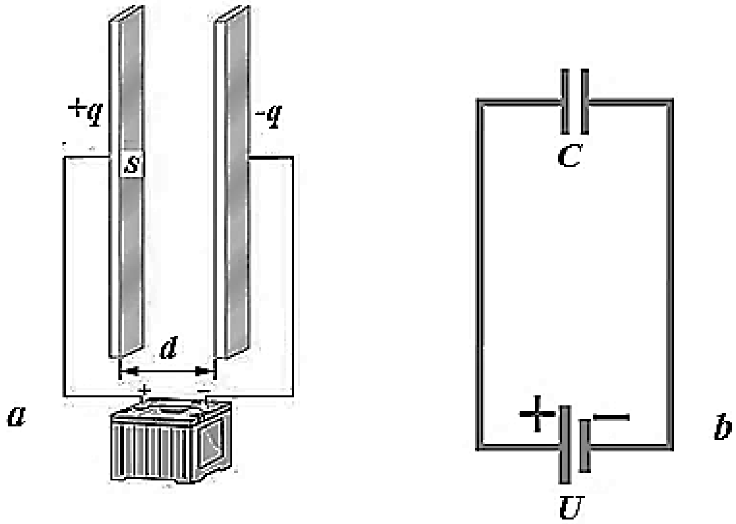


Fig. 10.15. Parallel-plate capacitor connected to a battery (*a*). The same electric circuit is shown using symbols (*b*)

The constant of proportionality, ***C***, in this relation is called the ***capacitance*** of the capacitor. Capacitance ***C*** is equal to the amount of charge ***q*** stored per volt. The unit of capacitance is coulombs per volt and this unit is called a ***farad*** (**1 F = 1 C/1 V**). A 1-farad capacitor would be able to store 1 coulomb (a very large amount of charge) with the application of only 1 volt. One farad is, thus, a very large capacitance. Most capacitors have capacitance in the range of 1 pF (picofarad = 10^{-12} F) to 1μF (microfarad = 10^{-6} F). The capacitance ***C*** does not depend on ***q*** or ***U***. It depends only on the size, shape, and relative position of the two capacitor plates, and also on the material that separates them. The capacitance of a parallel plate capacitor is given by formula:

$$\boldsymbol{C = \varepsilon \cdot \varepsilon_0 \frac{S}{d}}. \qquad (10.16)$$

where ***S*** is the area of one capacitor plate in square meters, and ***d*** is the distance between the plates in meters; the constant $\boldsymbol{\varepsilon_0}$ is the permittivity of free space and $\boldsymbol{\varepsilon}$ is a dielectric constant of a medium between the plates.

***Example 10.10*. Capacitor calculations.**

(a) Calculate the capacitance of a parallel-plate capacitor whose plates are 20 cm × 3.0 cm and are separated by a 1.0 mm air gap. (b) What is the charge on each plate if a 12 V battery is connected across the two plates? (c) What is the electric field between the plates? (d) Estimate the area of the plates needed to achieve a capacitance of 1 F, given the same air gap ***d***.

***Solution*.** The capacitance is found by using equation $C = \varepsilon_0 S/d$. The charge on each plate is obtained from the definition of capacitance $q = CU$. The electric field is uniform, so we can use equation for the magnitude $E = U/d$. In (d) we use equation $C = \varepsilon_0\, S/d$ again.

a) The area S = (20·10^{-2} m)·(3.0·10^{-2} m) = 6.0·10^{-3} m^2. The capacitance $\boldsymbol{C}$ is then

$$C = \varepsilon_0 \frac{S}{d} = (8.85 \cdot 10^{-12} \text{C}^2/\text{N} \cdot \text{m}^2 \frac{6.0 \cdot 10^{-3} \text{m}^2)}{1.0 \cdot 10^{-3} \text{m}} = 53\, pF.$$

b) The charge on each plate is $q = CU$ = (53·10^{-12} F)·(12 V) = 6.4·10^{-10} C.

c) For a uniform electric field, the magnitude of $\boldsymbol{E}$ is

$$E = \frac{U}{d} = \frac{12 \text{ V}}{1.0 \cdot 10^{-3} \text{m}} = 1.2 \cdot 10^4 \text{V/m}.$$

d) We solve for $\boldsymbol{S}$ in equation $\boldsymbol{C} = \varepsilon_0\ S/d$ and substitute $\boldsymbol{C}$ = 1.0 F and $\boldsymbol{d}$ = 1.0 mm to find that we need plates with an area

$$S = \frac{Cd}{\varepsilon_0} = \frac{(1F)(1.0 \cdot 10^{-3} \text{m})}{8.85 \cdot 10^{-12} \text{C}^2/\text{N} \cdot \text{m}^2} \approx 10^8 \text{m}^2.$$

NOTE. *This is the area of a square of* ***10^4*** *m =* ***10*** *km on a side!*

Example 10.11.**Capacitance of an axon.**

Do an order- of magnitude estimate for the capacitance of an axon 10 cm long of radius 10 μm. The thickness of the membrane is about 10^{-8} m, and the dielectric constant is about 3.

Solution. We model the membrane of an axon as a cylindrically shaped parallel-plate capacitor, with opposite charges on each side. The separation of the "plates" is the thickness of the membrane, d ~ 10^{-8} m. We first calculate the area of the cylinder

$$S = 2\pi rl = 6.28 \cdot 10^{-5} \text{ m} \cdot (0.1 \text{ m}) \approx 6 \cdot 10^{-6} \text{ m}^2.$$

and then can use equation $C = \varepsilon_0\ S/d$ to find the capacitance. The area $\boldsymbol{S}$ is the area of a cylinder of radius $\boldsymbol{r}$ and length $\boldsymbol{l}$:

The capacitance $\boldsymbol{C}$ is then

$$C = \varepsilon\varepsilon_0 \frac{S}{d} \approx 3 \cdot \left(8.85 \cdot 10^{-12} \text{C}^2/\text{N} \cdot \text{m}^2 \frac{6.0 \cdot 10^{-6} \text{m}^2}{10^{-8} \text{m}} \right) \approx 10^{-8} F.$$

10.7. THE EQUIVALENT CAPACITANCE

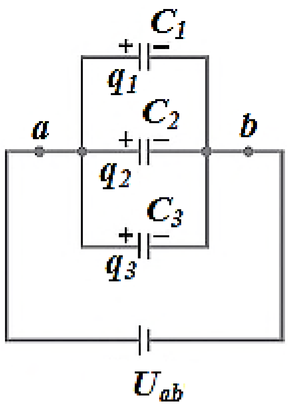


Fig. 10.16. Capacitors in parallel

Capacitors are found in many electric circuits. Capacitors can be connected together in various ways. Two common ways are in series, or in parallel. A circuit containing three capacitors connected in **parallel** is shown in fig. 10.16.

When capacitors are connected in **parallel**, the equivalent capacitance is the sum of the individual capacitances:

$$\boldsymbol{C_{eq} = C_1 + C_2 + C_3}. \qquad (10.17)$$

The net effect of connecting capacitors in parallel is thus to increase the capacitance. This makes sense because we are essentially increasing the area of the plates where charge can accumulate.

When capacitors are connected **in series** (fig. 10.17), the reciprocal of the equivalent capacitance equals the sum of the reciprocals of the individual capacitances:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}. \qquad (10.18)$$

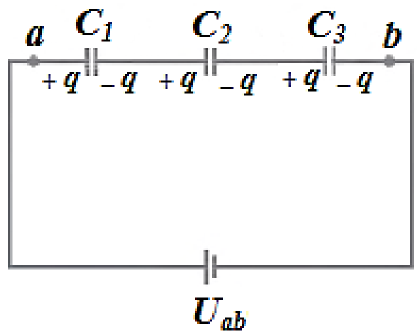


Fig. 10.17. Capacitors in series

10.8. ENERGY STORED IN A CAPACITOR

A charged capacitor stores electric energy. The energy stored in a capacitor is equal to the work done to charge the capacitor. Energy stored in a capacitor is electrical potential energy $\boldsymbol{W_{pot}}$, and it is thus related to the charge $\boldsymbol{q}$ and voltage $\boldsymbol{U}$ on the capacitor. From the definition of capacitance $\boldsymbol{C}$, the energy stored in a capacitor can be written in different forms:

$$\boldsymbol{W_{pot} = \frac{1}{2}qU = \frac{1}{2}CU^2 = \frac{q^2}{2C}}. \qquad (10.19)$$

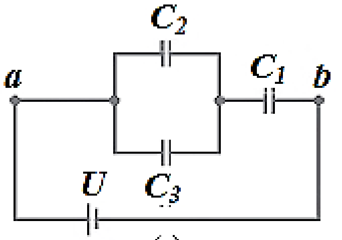
If we divide the stored energy by the volume of the capacitor, we find the electric energy per unit volume of capacitor of $\boldsymbol{C}$; this result is valid for any electric field:

$$\boldsymbol{w_{pot} = \text{electric energy density} = \frac{1}{2}\varepsilon\varepsilon_0 E^2}. \qquad (10.20)$$

***Example 10.12*. Equivalent capacitance.**

Determine the capacitance of a single capacitor that will have the same effect as the combination shown in fig. 10.18. Take $\boldsymbol{C_1 = C_2 = C_3 = C}$.

Solution. First we find the equivalent capacitance of $\boldsymbol{C_2}$ and $\boldsymbol{C_3}$ in parallel, and then consider that capacitance in series with $\boldsymbol{C_1}$. Capacitors $\boldsymbol{C_2}$ and $\boldsymbol{C_3}$ are connected in parallel, so they are equivalent to a single capacitor having capacitance: $\boldsymbol{C_{23} = C_2 + C_3 = 2C}$.



(a)

Fig. 10.18. Example 10.12

This $\boldsymbol{C_{23}}$ is in series with $\boldsymbol{C_1}$, fig. 10.18, so the equivalent capacitance of the entire circuit, $\boldsymbol{C_{eq}}$, is given by $\frac{1}{C_{\mathrm{eq}}} = \frac{1}{C_1} + \frac{1}{C_{23}} = \frac{1}{C} + \frac{1}{2C} = \frac{3}{2C}$.

Thus C_{eq} = 3/2 C.

NOTE. *Hence the equivalent capacitance of the entire combination is C_{eq} = 3/2 C.*

***Example 10.13*. Energy stored in a capacitor.**

A camera flash unit stores energy in a 150 μF capacitor at 200 V. (a) How much electric energy can be stored? (b) What is the power output if nearly all this energy is released in 1.0 ms?

Solution*.** We use equation for energy in the form $W_{\mathrm{pot}} = \frac{1}{2}CU^2$ because we are given ***C and ***U***.

The energy stored is $W_{\mathrm{pot}} = \frac{1}{2}CU^2 = \frac{1}{2}(150 \cdot 10^{-6} F)(200V)^2 = 3.0$ J.

If this energy is released in 1/1000 of a second, the power output is

$$P = W/t = (3.0\text{ J}) / (1.0 \cdot 10^{-3}\text{ s}) = 3000\text{ W}.$$

PROBLEMS

1. The membrane that surrounds a certain type of living cell has a surface area of and $5.0 \cdot 10^{-9}$ m^2 and a thickness of $1.0 \cdot 10^{-8}$ m. Assume that the membrane behaves like a parallel plate capacitor and has a dielectric constant of 5.0. (a) The potential on the outer surface of the membrane is +60.0 mV greater than that on the inside surface. How much charge resides on the outer surface? (b) If the charge in part (a) is due to K+ ions (charge +*e*), how many such ions are present on the outer surface? (Answer: a) $1.3 \cdot 10^{-12}$ C; b) $8.1 \cdot 10^6$)

2. A heart defibrillator delivers $4.00 \cdot 10^2$ J of energy by discharging a capacitor initially at $1.00 \cdot 10^4$ V. What is its capacitance? (Answer: 8.00 μF)

3. Find the total capacitance for three capacitors connected in series, given their individual capacitances are 1.000, 5.000, and 8.000 μF. (Answer: 0.755 μF)

Tests

1. SI unit of capacitance is:
a) Coulomb/Volt = Farad (F); b) F·V; c) F/A; d) F·N.

2. The dielectric with the permittivity **ε** filled the space between the plates of capacitor. Capacitance ***C*** of the capacitor is determined by:
a) $\varepsilon_0 \frac{S}{d}$; b) $\varepsilon_0 \varepsilon \frac{S}{d}$; c) $\varepsilon \frac{d}{S}$; d) $\varepsilon \frac{S}{d}$.

3. The capacitance of a parallel plane capacitor depends on:
a) the type of metal used;
b) the thickness of the plates;
c) the potential applied across the plates;
d) the separation between the plates.

4. The two plates of a capacitor hold +2500 μC and –2500 μC of charge, respectively, when the potential difference is 950 V. What is the capacitance?
a) $2.6 \cdot 10^{-6}$ F; b) $2.6 \cdot 10^{6}$ F; c) $3.8 \cdot 10^{-5}$ F; d) 2.6 μF.

5. Two parallel plates are separated by 2 cm. If the potential difference between them is 20 V, then the electric field between them is:
a) 100 N/C; b) 1000 N/C; c) 2000 N/C; d) zero.

6. A 12000 pF capacitor holds $28.0 \cdot 10^{-8}$ C of charge. What is the voltage across the capacitor?
a) 23.3 V; b) 2.33 V; c) 23 kV; d) none of the above.

7. A parallel plate capacitor is charged from a battery. After charging, the battery is disconnected. Which of the following increases, when the plates of the capacitor are moved apart?
a) charge; b) capacitance; c) potential; d) none of these.

8. What is the equivalent capacitance of the combination shown in fig. 10.19?
a) C; b) 2C; c) 4C; d) C/4.

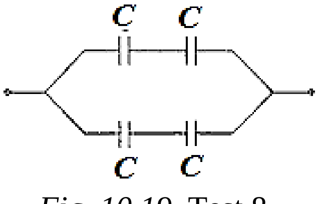


Fig. 10.19. Test 8

9. In fig. 10.20, what is the effective capacitance between points ***P*** and ***Q***?
a) 6/11 μF; b) 2/11 μF;
c) 24/17 μF; d) none of these.

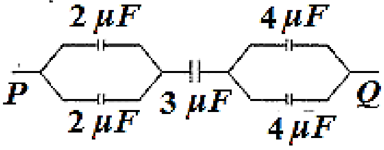


Fig. 10.20. Test 9

10.9. AN ELECTRIC CURRENT

In the previous Chapters we have been studying static electricity: electric charges at rest. In this Chapter we begin our study of charges in motion. It is known the charges can move in electric field.

A directed flow of charged particles is called ***an electric current***. There are two conditions for an electric current existence:

1. The presence of free charges.
2. The presence of an electric field.

The materials are divided into two categories: conductors and insulators (dielectrics). Bodies which allow the charges to pass through are called ***conductors***, e. g. metals, human body, Earth etc. Bodies which do not allow the charges to pass through are called ***insulators***, e. g. glass, mica, ebonite, plastic etc. Nearly all natural materials fall into one or the other of these two very distinct categories. However, a few materials (notably silicon and germanium) fall into an intermediate category known as ***semiconductors***.

From the atomic point of view, the electrons in an insulating material are bound very tightly to the nuclei. In a good conductor, on the other hand, some of the electrons are bound very loosely and can move about freely within the material (although they cannot *leave* the object easily) and are often referred to as *free electrons* or *conduction electrons.* In a semiconductor, there are very few free electrons, and in an insulator, almost none.

10.10. DIRECT CURRENT. OHM'S LAW. RESISTANCE

To produce an electric current it is necessary to apply a potential difference at conducting material. One way of producing a potential difference along a wire is to connect its ends to the opposite terminals of a battery (fig. 10.21).

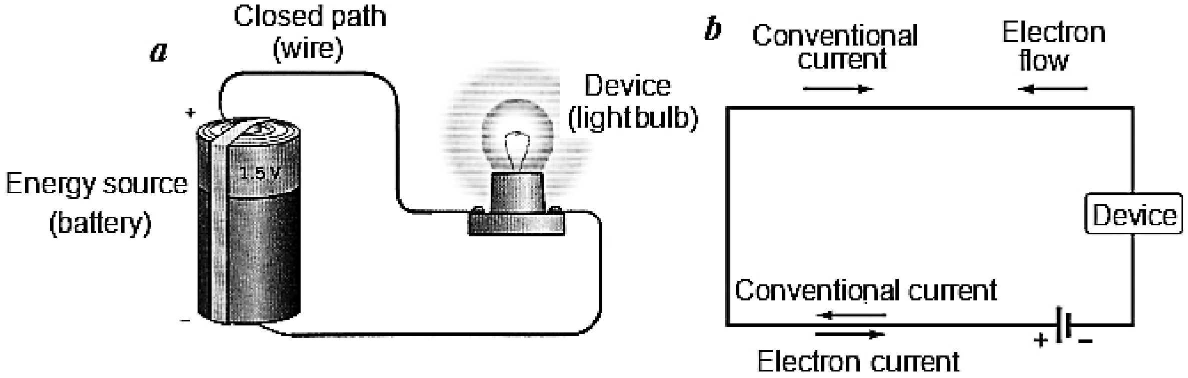


Fig. 10.21. A simple electric circuit (*a*). Schematic drawing of the same circuit (*b*)

On any diagram of a circuit a battery is symbolized by the symbol .

The electric current is defined as the net amount of charge flowing through cross section of a conductor per unit time. Thus, the average current $\boldsymbol{I}$ is defined as:

$$\boldsymbol{I} = \frac{\Delta \boldsymbol{q}}{\Delta \boldsymbol{t}}, \tag{10.21}$$

where $\Delta \boldsymbol{q}$ is the amount of charge that passes through the conductor during the time interval $\Delta \boldsymbol{t}$.

Electric current is measured in **amperes** (abbreviated ***A***). When the flow of charge past any cross section is 1 coulomb per second, the current is 1 ampere. Thus, 1 A = 1 C/1 s. Ampere is a large unit for current. In practice smaller units are often used, such as the milliampere (l mA = 10^{-3} A) and microampere (1 μA = 10^{-6} A).

***Example 10.14*. Current is flow of charge.**

A steady current of 2.5 A exists in a wire for 4.0 min. (a) How much total charge passed by a given point in the circuit during those 4.0 min? (b) How many electrons would this be?

Solution. Current is flow of charge per unit time, $I = \Delta q/\Delta t$, so the amount of charge passing a point is the product of the current and the time interval. To get the number of electrons (b), we divide the total charge by the charge on one electron.

a) Since the current was 2.5 A, or 2.5 C/s, then in 4.0 min (= 240 s) the total charge that flowed past a given point in the wire was

$$\Delta q = I\Delta t = (2.5\ \text{C/s})(240\ \text{s}) = 600\ \text{C}.$$

b) The charge on one electron is $1.60 \cdot 10^{-19}$ C, so 600 C would consist of

$$\frac{600\ \text{C}}{1.60 \cdot 10^{-19}\ \text{C/electron}} = 3.8 \cdot 10^{21}\ \text{electrons}.$$

Current is a scalar quantity. The arrows in fig. 10.21, *b* do not indicate vectors: they merely show direction of flow along a conductor, not a direction in space. ***The direction of the current is*** taken to be ***the direction of the flow of positive charge***, even if the actual charge carriers are negative and move in the opposite direction.

George Simon Ohm established the relationship between potential difference and current, which is known as Ohm's Law. The law states that the steady current $\boldsymbol{I}$ flowing through a conductor is directly proportional to the potential difference $\boldsymbol{U}$ between the two ends of the conductor:

$$I = \frac{U}{R}, \tag{10.22}$$

where the constant $\boldsymbol{R}$ is known as the resistance of a conductor. The SI unit for resistance is called the ohm and is abbreviated Ω: 1 Ω = 1 V/1 A.

In a circuit diagram a resistor and a resistance is represented by  the symbol. Resistance $\boldsymbol{R}$ depends on the conductor geometrical size (fig. 10.22), the material the conductor is made of, and the temperature of the conductor.

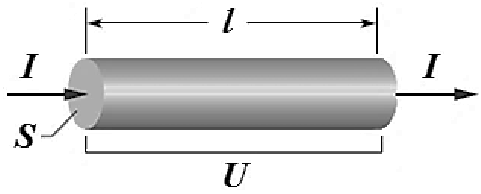


Fig. 10.22. Current $\boldsymbol{I}$ is driven by a potential difference $\boldsymbol{U}$ applied between the ends of a conductor of length $\boldsymbol{l}$ and cross section $\boldsymbol{S}$

It is found experimentally that the resistance of a conductor $\boldsymbol{R}$ is directly proportional to the length of the conductor $\boldsymbol{l}$ and is inversely proportional to its area of cross section $\boldsymbol{S}$:

$$\boldsymbol{R} = \frac{\rho \boldsymbol{l}}{\boldsymbol{S}}, \tag{10.23}$$

where **ρ**, the constant of proportionality, is called ***specific resistance*** or ***electrical resistivity*** of the material of the conductor and depends on the material used.

If $\boldsymbol{l}$ = 1 m, $\boldsymbol{S}$ = 1 m^2, then **ρ** = $\boldsymbol{R}$. The electrical resistivity **ρ** of a material is defined as the resistance $\boldsymbol{R}$ offered to current flow by a conductor of unit length $\boldsymbol{l}$ having unit area of cross section $\boldsymbol{S}$. The unit of **ρ** is **Ω·m**. It is a constant for a particular material. The electrical resistivity **ρ** of a material depends somewhat on temperature.

The reciprocal of electrical resistivity **ρ**, is called ***electrical conductivity*** **σ**:

$$\boldsymbol{\sigma} = \frac{\mathbf{1}}{\boldsymbol{\rho}}. \tag{10.24}$$

The unit of conductivity **σ** is $\mathbf{\Omega^{-1}\ m^{-1}}$.

For most conductors, a temperature increase causes an increase in resistance. If the temperature change is not too great, the resistance $\boldsymbol{R}$ of metals usually increases nearly linearly with temperature. An empirical relationship for the temperature dependence of the resistance $\boldsymbol{R}$ of metals is given by formula:

$$\boldsymbol{R} = \boldsymbol{R}_0[1 + \alpha(\boldsymbol{T} - \boldsymbol{T}_0)], \tag{10.25}$$

where $\boldsymbol{R}_0$ is the resistance at some reference temperature $\boldsymbol{T}_0$ (such as 0 °C or 20 °C); $\boldsymbol{R}$ is the resistance at a temperature $\boldsymbol{T}$; **α** is the temperature coefficient of resistance.

Resistance $\boldsymbol{R}$ is a property of an object, whereas resistivity **ρ** is a property of a material.

***Example 10.15*. Current is flow of charge.**

A steady current of 2.5 A exists in a wire for 4.0 min. (a) How much total charge passed by a given point in the circuit during those 4.0 min? (b) How many electrons would this be?

Solution. Current is flow of charge per unit time, $I = \Delta q/\Delta t$, so the amount of charge passing a point is the product of the current and the time interval. To get the number of electrons (b), we divide the total charge by the charge on one electron.

a) Since the current was 2.5 A, or 2.5 C/s, then in 4.0 min (= 240 s) the total charge that flowed past a given point in the wire was

$$\Delta q = I\Delta t = (2.5\ \text{C/s})(240\ \text{s}) = 600\ \text{C}.$$

b) The charge on one electron is $1.60 \cdot 10^{-19}$ C, so 600 C would consist of

$$\frac{600\ \text{C}}{1.60 \cdot 10^{-19}\ \text{C/electron}} = 3.8 \cdot 10^{21}\ \text{electrons}.$$

***Example 10.16*. Speaker wires.**

Suppose you want to connect your stereo to remote speakers (fig. 10.23). (a) If each wire must be 20 m long, what diameter copper wire should you use to keep the resistance less than 0.10 Ω per wire? (b) If the current to each speaker is 4.0 A, what is the potential difference, or voltage drop, across each wire? The resistivity of copper is $1.68 \cdot 10^{-8}$ Ω·m.

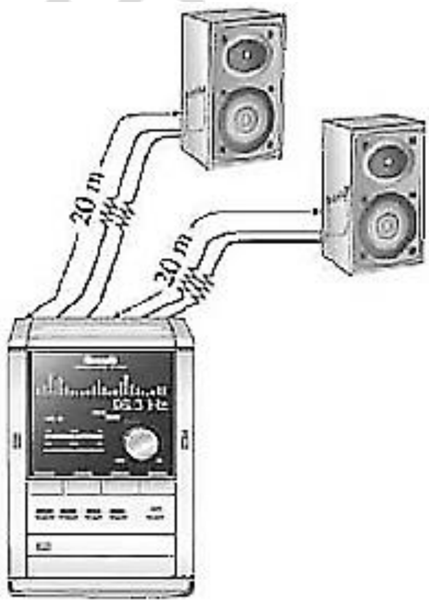


Fig. 10.23. Example 10.16

Solution. We use equation $R = \dfrac{\rho l}{S}$ to get the area ***S***, from which we can calculate the wire's radius using $S = \pi r^2$. The diameter is $2r$. In (b) we can use Ohm's law, $U = IR$.

$$S = \frac{\rho l}{R} = \frac{(1.68 \cdot 10^{-8}\ \Omega \cdot \text{m}) \cdot (20\ \text{m})}{0.10\ \Omega} = 3.4 \cdot 10^{-6}\ \text{m}^2.$$

The cross-sectional area ***S*** of a circular wire is $S = \pi r^2$. The radius must then be at least $r = \sqrt{\dfrac{S}{\pi}} = 1.04 \cdot 10^{-3}\ \text{m} = 1.04\ \text{mm}.$

The diameter is twice the radius and so must be at least $2r = 2.1$ mm.

b) From $V = IR$ we find that the voltage drop across each wire is

$$U = IR = 4.0\ \text{A} \cdot 0{,}10\ \Omega = 0.40\ \text{V}.$$

NOTE. *The voltage drop across the wires reduces the voltage that reaches the speakers from the stereo amplifier, thus reducing the sound level a bit.*

***Example 10.17*. Stretching changes resistance.**

Suppose a wire of resistance ***R*** could be stretched uniformly until it was twice its original length. What would happen to its resistance?

Solution. If the length ***l*** doubles, then the cross-sectional area ***S*** is halved, because the volume ($V = Sl$) of the wire remains the same. From $R = \frac{\rho l}{S}$ we see that the resistance would increase by a factor of four $2/\frac{1}{2} = 4$.

Example 10.18.

A uniform copper wire, having mass of $2.23 \cdot 10^{-3}$ kg carries a current of 1 A, has potential difference of 1.7 V across its ends. Find its length and area of cross section. The wire is now uniformly stretched to double its length, find the new resistance. Density of copper is $8.92 \cdot 10^{3}$ kg·m^{-3} and its resistivity is $1.7 \cdot 10^{-8}$ Ω·m.

Solution. Suppose ***l*** and ***S*** be the length and area of cross section of the given copper wire. Volume ***V*** of the wire is ***V = Sl***

$$V = Sl = \frac{\text{mass}}{\text{density}} = \frac{2.23 \cdot 10^{-3}\text{kg}}{8.92 \cdot 10^{3}\text{kg} \cdot \text{m}^{-3}}$$

or
$$Sl = \frac{10^{-6}}{4}\text{m}^3.$$

Also, resistance of the copper wire is given by $\frac{l}{S} = 10^8 \text{m}^{-1}$,

or
$$R = \frac{\rho l}{S} = \frac{U}{I} = \frac{1.7V}{1A} = 1.7\ \Omega.$$

Combining $Sl = \frac{10^{-6}}{4}\text{m}^3$ and we get $\frac{l}{S} = \frac{R}{\rho} = \frac{1.7\ \Omega}{1.7 \cdot 10^{-8}\ \Omega \cdot \text{m}} = 10^8\ \text{m}^{-1}$.

Also, $S = 5 \cdot 10^{-8}$ m^2. $l^2 = \frac{10^{-6}}{4}\text{m}^3 \cdot 10^8 \text{m}^1 = 25\ \text{m}^2$ or $l = 5$ m.

Since volume remains constant, $V = Sl = S_1 l_1$, where $l_1 = 2l$ and $S_1 = \frac{S}{2}$.

Then $R_1 = \frac{\rho l_1}{S_1} = \rho\frac{2l}{S/2} = 4\rho\frac{l}{S} = 4R$, where $R = \frac{\rho l}{S}$

$$R_1 = \frac{4 \cdot 1.7 \cdot 10^{-8}\ \Omega \cdot \text{m} \cdot 5\ \text{m}}{5 \cdot 10^{-8}\ \text{m}^2} = 6.8\ \Omega.$$

***Example 10.19*. Resistance thermometer.**

The variation in electrical resistance with temperature can be used to make precise temperature measurements. Platinum is commonly used since it is relatively free from corrosive effects and has a high melting point. Suppose at 20.0 °C the resistance of a platinum resistance thermometer is 164.2 Ω. When placed in a particular solution, the resistance is 187.4 Ω. The temperature coefficient of resistivity of platinum **α** is equal to $3.927 \cdot 10^{-3}$ $(C^o)^{-1}$. What is the temperature of this solution?

Solution. Since the resistance ***R*** is directly proportional to the resistivity **ρ**, we can combine equation $R = \frac{\rho \cdot l}{S}$ with equation $R = R_0[1 + \alpha(T - T_0)]$ to find ***R*** as a function of temperature ***T***, and then solve that equation for ***T***.

We use the equation $R = R_0[1 + \alpha(T - T_0)]$.

Here $R_0 = \rho_0 l/S$ is the resistance of the wire at T_0 = 20.0 °C. We solve this equation for ***T*** and find

$$T = T_0 + \frac{R - R_0}{\alpha R_0} = 20.0\ ^\circ\text{C} + \frac{187.4\ \Omega - 164.2\ \Omega}{3.927 \cdot 10^{-3}(\text{C}^o)^{-1}(164.2\ \Omega)} = 56.0\ ^\circ\text{C}.$$

NOTE. *Resistance thermometers have the advantage that they can be used at very high or low temperatures where gas or liquid thermometers would be useless.*

10.11. RESISTORS IN SERIES AND IN PARALLEL

There are active and passive elements of electrical circuit. Active elements can generate energy (voltage and current sources, batteries), passive ones cannot generate energy (resistors, capacitors and inductors). A resistor is a circuit element that dissipates electrical energy (usually as heat). Devices that are modeled by resistors: light bulbs, heating elements (stoves, heaters, etc.), long wires.

When two or more resistors are connected end to end as shown in fig. 10.24, they are said to be connected in series.

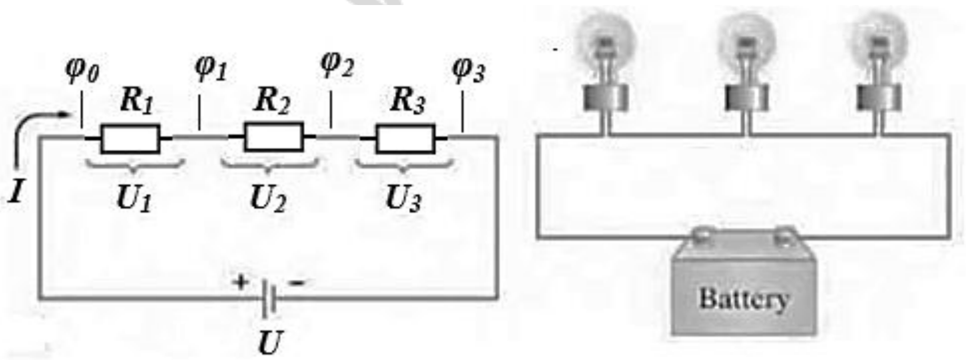


Fig. 10.24. Resistors in series

Any charge that passes through $\boldsymbol{R_1}$ in fig. 10.24 will also pass through $\boldsymbol{R_2}$ and then $\boldsymbol{R_3}$. Hence the same current ***I*** passes through each resistor $\boldsymbol{I = I_1 = I_2 = I_3}$.

We let ***U*** represent the potential difference (voltage) across all three resistors in fig. 10.24. We assume all other resistance in the circuit can be ignored, so U equals the terminal voltage supplied by the battery. We let U_1, U_2 and U_3 be the potential differences across each of the resistors, R_1, R_2 and R_3, respectively, as shown in fig. 10.24. From Ohm's Law $U = IR$ one can write $U_1 = IR_1$, $U_2 = IR_2$ and $U_3 = IR_3$. Because the resistors are connected end to end the total voltage U is equal to the sum of the voltages across each resistor:

$$U_1 + U_2 + U_3 = \varphi_0 - \varphi_1 + \varphi_1 - \varphi_2 + \varphi_2 - \varphi_3 = \varphi_0 - \varphi_3 = U.$$

Thus,

$$U = U_1 + U_2 + U_3 = IR_1 + IR_2 + IR_3. \qquad (10.26)$$

Let's determine the equivalent (or effective) single resistance R_{eq} of the series combination. The equivalent single resistance R_{eq} is related to U by

$$U = IR_{eq}.$$

Hence, $IR_{eq} = IR_1 + IR_2 + IR_3$,

or

$$R_{eq} = R_1 + R_2 + R_3. \qquad (10.27)$$

Thus, ***the equivalent resistance of a number of resistors in series connection is equal to the sum of the resistance of individual resistors.***

Consider resistors of resistances R_1, R_2 and R_3 are connected in parallel, so that the current from the source splits into separate branches as shown in fig. 10.25.

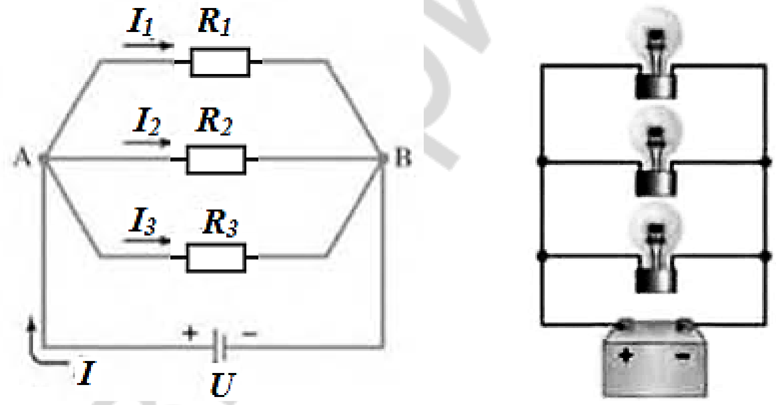


Fig. 10.25. Resistors in parallel

In parallel circuit the total current I that leaves the battery splits into three separate paths. Let I_1, I_2 and I_3 be the currents through each of the resistors, R_1, R_2 and R_3, respectively. Because electric charge is conserved, the current I flowing into a junction A (where the different wires or conductors meet) must equal the current flowing out of the junction. Thus,

$$I = I_1 + I_2 + I_3. \qquad (10.28)$$

When resistors are in parallel, the potential difference U across each resistor is the same ($U = U_1 = U_2 = U_3$). Applying Ohm's Law to each resistor one can write:

$$I_1 = \frac{U}{R_1}, \qquad I_2 = \frac{U}{R_2}, \qquad I_3 = \frac{U}{R_3}. \qquad (10.29)$$

Let's determined what single resistor $\boldsymbol{R}_{\mathbf{eq}}$ will draw the same current $\boldsymbol{I}$ as these three resistances in parallel:

$$\boldsymbol{I} = \frac{\boldsymbol{U}}{\boldsymbol{R}_{\mathbf{eq}}}. \tag{10.30}$$

Thus,

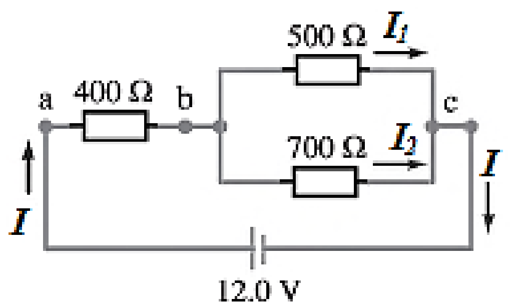
$$\frac{\boldsymbol{U}}{\boldsymbol{R}_{\mathbf{eq}}} = \frac{\boldsymbol{U}}{\boldsymbol{R}_1} + \frac{\boldsymbol{U}}{\boldsymbol{R}_2} + \frac{\boldsymbol{U}}{\boldsymbol{R}_3}, \tag{10.31}$$

or

$$\frac{\mathbf{1}}{\boldsymbol{R}_{\mathbf{eq}}} = \frac{\mathbf{1}}{\boldsymbol{R}_1} + \frac{\mathbf{1}}{\boldsymbol{R}_2} + \frac{\mathbf{1}}{\boldsymbol{R}_3}. \tag{10.32}$$

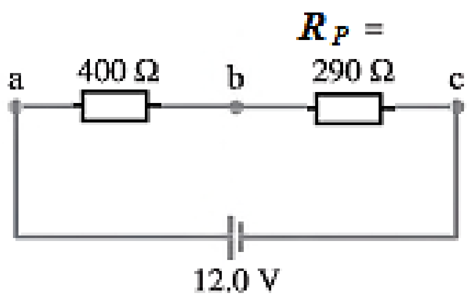
Thus, ***when a number of resistors are connected in parallel, the sum of the reciprocal of the resistance of the individual resistors is equal to the reciprocal of the effective resistance of the combination.***

***Example 10.20*. Circuit with series and parallel resistors.**
How much current is drawn from the battery shown in fig. 10.26, *a*?



a

Fig. 10.26. Circuit for Example 10.20 (*a*).



b

Fig. 10.26. (*b*) Equivalent circuit, showing the equivalent resistance of 290 Ω for the two parallel resistors in fig. 10.6, *a*

Solution. The current $\boldsymbol{I}$ that flows out of the battery all passes through the 400 Ω resistor, but then it splits into $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ passing through the 500 Ω and 700 Ω resistors. The latter two resistors are in parallel with each other. We look for something that we already know how to treat. So let's start by finding the equivalent resistance, $\boldsymbol{R}_p$, of the parallel resistors, 500 Ω and 700 Ω. Then we can consider this $\boldsymbol{R}_p$ to be in series with the 400 Ω resistor.

The equivalent resistance, $\boldsymbol{R}_p$, of the 500 Ω and 700 Ω resistors in parallel is given by

$$\frac{1}{R_p} = \frac{1}{500\ \Omega} + \frac{1}{700\ \Omega} = 0.0020\ \Omega^{-1} + 0.0014\ \Omega^{-1} = 0.0034\ \Omega^{-1}.$$

This is $\boldsymbol{l}/\boldsymbol{R}_p$, so we take the reciprocal to find $\boldsymbol{R}_p$. It is a common mistake to forget to take this reciprocal. Notice that the units of reciprocal ohms, Ω^{-1}, are a reminder.

Thus $$R_p = \frac{1}{0.0034\ \Omega^{-1}} = 290\ \Omega.$$

This 290 Ω is the equivalent resistance of the two parallel resistors, and is in series with the 400 Ω resistor as shown in the equivalent circuit of fig. 10.26, *b*.

To find the total equivalent resistance $\boldsymbol{R_{eq}}$, we add the 400 Ω and 290 Ω resistances together, since they are in series, and find

$$\boldsymbol{R_{eq}} = 400\ \Omega + 290\ \Omega = 690\ \Omega.$$

The total current flowing from the battery is then

$$I = \frac{U}{R_{eq}} = \frac{12.0\ \text{V}}{690\ \Omega} = 0.0174\ \text{A} \approx 17\ \text{mA}.$$

10.12. ELECTRIC ENERGY AND ELECTRIC POWER

If $\boldsymbol{I}$ is the current flowing through a conductor of resistance $\boldsymbol{R}$ in time $\boldsymbol{\Delta t}$, then the quantity of charge flowing is $\boldsymbol{\Delta q = I\Delta t}$. If the charge $\boldsymbol{\Delta q}$ flows between two points having a potential difference $\boldsymbol{U}$, then the work $\boldsymbol{\Delta W}$ done in moving the charge is the product of potential difference $\boldsymbol{U}$ and the charge $\boldsymbol{\Delta q}$:

$$\boldsymbol{\Delta W = U\cdot\Delta q = U\cdot I\Delta t.} \tag{10.33}$$

The unit of the work $\boldsymbol{W}$ is Joule (***J***).

Then, electric power $\boldsymbol{P}$ is defined as the rate of doing electric work and is equal to the product of current $\boldsymbol{I}$ and voltage $\boldsymbol{U}$:

$$\boldsymbol{P = \frac{\Delta W}{\Delta t} = IU.} \tag{10.34}$$

For resistors the electric power $\boldsymbol{P}$ can be written as:

$$\boldsymbol{P = IU = I(IR) = I^2R = \left(\frac{U}{R}\right)U = \frac{U^2}{R}.} \tag{10.35}$$

When resistors are in parallel, the potential difference $\boldsymbol{U}$ across each resistor is the same and electric power $\boldsymbol{P}$ can be found as: $\boldsymbol{P = \frac{U^2}{R}}$.

When resistors are connected in series, the same current $\boldsymbol{I}$ passes through each resistor and electric power $\boldsymbol{P}$ can be found as: $\boldsymbol{P = I^2R}$.

The SI unit of electric power $\boldsymbol{P}$ is the Watt (W): **1 W = 1 J/s**.

A resistor dissipates power when a current passes through it. The energy is released in the form of heat. For a steady current $\boldsymbol{I}$, the amount of heat $\boldsymbol{Q}$ produced in time $\boldsymbol{\Delta t}$ is equal to the work $\boldsymbol{\Delta A}$ done in moving the charges:

$$\boldsymbol{Q = P\Delta t = UI\Delta t.} \tag{10.36}$$

For a resistance $\boldsymbol{R}$, $$\boldsymbol{Q = I^2R\Delta t} \tag{10.37}$$

$$\boldsymbol{Q = \frac{U^2}{R}\Delta t.} \tag{10.38}$$

The above relations were experimentally verified by Joule and are known as Joule's Law of heating. By equation (10.37) Joule's law implies that when current flows through a conductor, the heat produced $\boldsymbol{Q}$ is directly proportional to the square of the current $\boldsymbol{I}$, directly proportional to resistance $\boldsymbol{R}$ of a conductor and the time $\boldsymbol{\Delta t}$ of passage of current. Also by equation (10.38), the heat produced $\boldsymbol{Q}$ is directly proportional to the square of the voltage $\boldsymbol{U}$ and inversely proportional to resistance $\boldsymbol{R}$ for a given $\boldsymbol{U}$. This is also known as Joule heat that is dissipated in $\boldsymbol{R}$.

***Example 10.21*. Headlights.**

Calculate the resistance of a 40-W automobileheadlight designed for 12 V.

Solution. We solve equation $P = U^2/R$ for $\boldsymbol{R}$:

$$R = \frac{U^2}{P} = \frac{(12\text{ V})^2}{(40\text{ W})} = 3.6\ \Omega.$$

Fig. 10.27. Example 10.21

NOTE. *This is the resistance when the bulb is burning brightly at 40 W. When the bulb is cold, the resistance is much lower, as we saw in equation (10.25). Since the current is high when the resistance is low, lightbulbs burn out most often when first turned on.*

***Example 10.22*.**

A 100 W bulb and a 400 W bulb are joined in parallel to the mains. Which bulb will draw more current?

Solution. Let $\boldsymbol{U}$ be the voltage of the mains and $\boldsymbol{I_1}$, $\boldsymbol{I_2}$ are the currents through the two bulbs. In case of parallel combination of the bulbs, $\boldsymbol{U}$ is the same for these two bulbs: $P_1 = I_1 \cdot U$ and $P_2 = I_2 \cdot U$,

or
$$\frac{I_1}{I_2} = \frac{P_1}{P_2} = \frac{100\text{ W}}{400\text{ W}} = \frac{1}{4}$$

$$\frac{I_1}{I_2} = \frac{1}{4} \quad \text{or} \quad I_1 < I_2.$$

NOTE. *400 W bulb will draw more current.*

***Example 10.23*.**

An electric kettle draws a current of 10 A when connected to the 230 V mains supply. Calculate: (a) the power of the kettle; (b) the energy produced in 5 minutes; (c) the rise in temperature if all the energy produced in 5 minutes is used to heat 2 kg of water. (Specific heat capacity of water c = 4200 J kg^{-1} K^{-1}.)

Solution. We use equation $P = IU$ to calculate power $\boldsymbol{P}$ of the kettle:

a) $P = IU = 10\text{ A} \times 230\text{ V} = 2300\text{ W} = 2.3\text{ kW}$.

b) Heat energy produced in 5 minutes: $Q = Pt = 2300\text{ W} \times 5 \times 60\text{ s} = 690\,000\text{ J}$.

c) Heat $\boldsymbol{Q}$ produced is absorbed by water: Q = energy gained by water. We solve equation $Q = mc\Delta T$ for $\boldsymbol{\Delta T}$:

$$\Delta T = \frac{Q}{mc} = \frac{690\ 000\ \text{J}}{2\ \text{kg} \cdot 4200\ \text{J} \cdot \text{kg}^{-1}\text{K}^{-1}} = 82.1\ \text{K}.$$

NOTE. *Rise in temperature of the water is 82.1 K.*

10.13. ELECTROMOTIVE FORCE. OHM'S LAW FOR A COMPLETE CIRCUIT

To maintain a steady current, there must be a device (such as a battery or an electric generator) in the circuit wherein the potential rises along the direction of the current. For the potential to rise along the direction of the current there must be a source which converts some form of nonelectric energy (chemical, mechanical, or light, for example) to electrical energy. Such a device is called a ***source*** of ***electromotive force*** or of ***emf***. An ***emf*** device is a (charge pump) device that maintains a constant potential difference between a pair of terminals. The symbol **ε** is usually used for ***emf***. The ***emf*** **ε** of a device is the work $\boldsymbol{A_f}$ per unit charge $\boldsymbol{q}$ that the nonelectrostatic force does in moving charge from low potential terminal to high potential terminal:

$$\boldsymbol{\varepsilon = \frac{A_f}{q}}. \tag{10.39}$$

The unit of the ***emf*** = **ε** is Volt (***V***).

A battery itself has some resistance, which is called its internal resistance: it is usually designated ***r***. A real battery is modeled as if it were a perfect ***emf*** **ε** in series with a resistor ***r*** as shown in fig. 10.28. Since this resistance ***r*** is inside the battery one can never separate it from the battery.

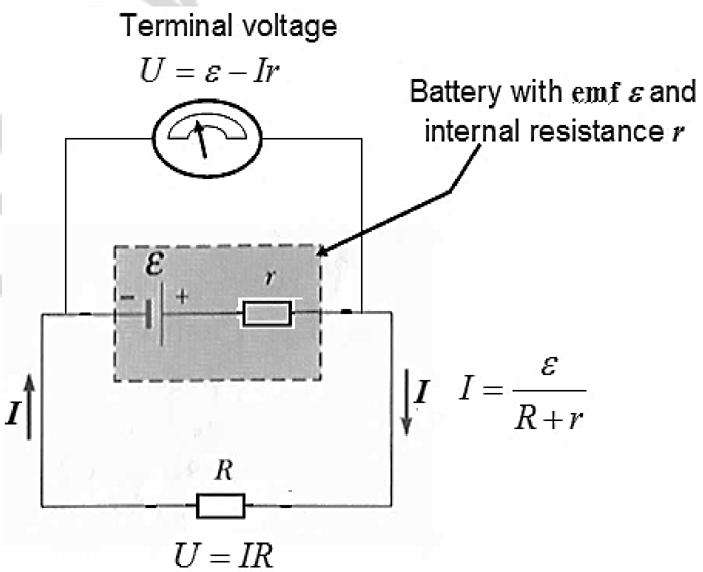


Fig. 10.28. A battery in a complete circuit

The work A_f that the nonelectrostatic force does in moving charge from low potential terminal to high potential terminal is given by:

$$A_f = \varepsilon \cdot q = \varepsilon (I \cdot t). \qquad (10.40)$$

The work A_f can be also written as (if all work A_f is converted into the heating of the conductor $A = Q$):

$$A_f = I^2 (R + r) \cdot t. \qquad (10.41)$$

Comparing equations (10.20) and (10.21) gives:

$$\varepsilon = I(R + r). \qquad (10.42)$$

For a complete circuit, Ohm's Law assumes the following form:

$$I = \frac{\varepsilon}{R + r}. \qquad (10.43)$$

where $R_t = R + r$ is the total resistance of the entire circuit, which is equal to the sum of the external resistance R of the circuit and the internal resistance r of the source of ***emf***.

When connected to a load resistance R, so that a current I flows through the circuit there is an internal drop in voltage equal to ($I \cdot r$). Thus the terminal voltage of the battery is therefore:

$$U = \varepsilon - Ir. \qquad (10.44)$$

The potential drop across the external resistance R can be written as:

$$U = IR. \qquad (10.45)$$

The internal resistance r of the source of ***emf*** is low. When the external resistance R of the circuit drops ($R \to 0$), current becomes very high and can cause damage. The largest amount of current in circuit is called the ***short-circuit*** current I_{sh}:

$$I_{sh} = \frac{\varepsilon}{r}. \qquad (10.46)$$

***Example 10.24*. Battery with internal resistance.**

A 65.0 Ω resistor is connected to the terminals of a battery whose ***emf*** is 12.0 V and whose internal resistance is 0.5 Ω. Calculate: (a) the current in the circuit, (b) the terminal voltage of the battery, U_{ab}, (c) the power dissipated in the resistor R and in the battery's internal resistance r.

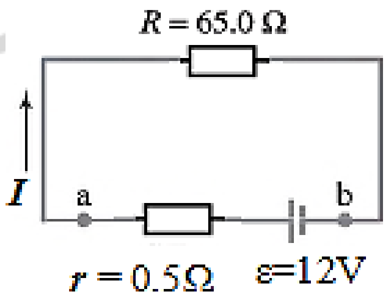


Fig. 10.29. Example 24

Solution. We first consider the battery as a whole, which is shown in fig. 10.29 as an ***emf*** ε and internal resistance r between points a and b. Then we apply $U = IR$ to the circuit itself.

a) From the equation relating ***emf*** **ε** to terminal voltage, we have $\boldsymbol{U_{ab}} = \boldsymbol{\varepsilon} - \boldsymbol{Ir}$. We apply Ohm's law to this battery and the resistance ***R*** of the circuit: $U_{ab} = IR$. Hence $IR = \varepsilon - Ir$ or $\varepsilon = I(R + r)$, and so

$$P_R = I^2R = (0.183\ \text{A})^2(65.0\ \Omega) = 2.18\ \text{W},$$

b) The terminal voltage is

$$U_{ab} = \varepsilon - Ir = 12.0\ \text{V} - (0.183\ \text{A})(0.5\ \Omega) = 11.9\ \text{V}.$$

c) The power dissipated in ***R*** is $P_R = I^2R = (0.183\ \text{A})^2(65.0\ \Omega) = 2.18\ \text{W}$, and in ***r*** is $P_r = I^2r = (0.183\ \text{A})^2(0.5\ \Omega) = 0.02\ \text{W}$.

10.14. ALTERNATING CURRENT

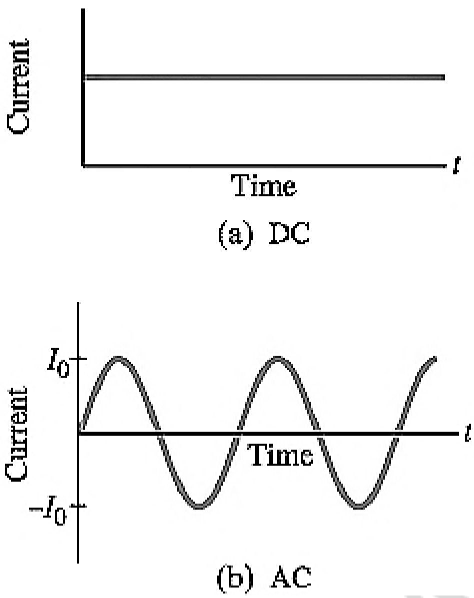


Fig. 10.30. (*a*) Direct current, (*b*) Alternating current

When a battery is connected to a circuit, the current moves steadily in one direction. This is called a **direct current,** or **dc.** Electric generators at electric power plants, however, produce **alternating current,** or **ac**. Sometimes capital letters are used, **DC** and **AC**. An alternating current reverses direction many times per second and is commonly sinusoidal, as shown in fig. 10.30. The charges in a wire first move in one direction and then in the other. The current supplied to homes and businesses by electric companies is ac throughout virtually the entire world. It is easier and cheaper to produce as well as transmit AC than DC.

An alternating current is one that changes continuously in magnitude and periodically in direction. The alternating currents varying according to harmonic law have the most important practical significance. It is represented by a sine curve (fig. 10.30, *b*). The mathematical form of an alternating voltage as a function of time is:

$$\boldsymbol{U = U_0 \sin 2\pi f t = U_0 \sin \omega t}. \tag{10.47}$$

The voltage oscillates between $+\boldsymbol{U_0}$ and $-\boldsymbol{U_0}$, and $\boldsymbol{U_0}$ is referred to as the ***peak voltage***. ***U*** is the instantaneous value of voltage at an instant of time ***t***. The frequency ***f*** is the number of complete oscillations made per second (measured in Hertz, the unit "hertz" means cycles per second 1/sec); and $\boldsymbol{\omega = 2\pi f}$ — is angular frequency (radians/sec). In many countries, $f = 50$ Hz is used.

From equation $\boldsymbol{U = IR}$, if a voltage $\boldsymbol{U}$ exists across a resistance $\boldsymbol{R}$, then the current $\boldsymbol{I}$ through the resistance is

$$\boldsymbol{I = \frac{U}{R} = \frac{U_0}{R}\sin\omega t = I_0\sin\omega t.} \qquad (10.48)$$

The quantity $\boldsymbol{I_0 = U_0/R}$ is the ***peak current***. The current is considered positive when the electrons flow in one direction and negative when they flow in the opposite direction. It is clear from fig. 10.30, *b* that an alternating current is as often positive as it is negative. Thus, the average current is zero. This does not mean, however, that no power is needed or that no heat is produced in a resistor. Electrons do move back and forth, and do produce heat. Indeed, the power transformed in a resistance $\boldsymbol{R}$ at any instant is

$$\boldsymbol{P = I^2R = I_0^2R\sin^2\omega t.} \qquad (10.49)$$

Because the current is squared, we see that the power is always positive, as graphed in fig. 10.31. The quantity $\boldsymbol{\sin^2\omega t}$ varies between 0 and 1; and it is not too difficult to show that its average value is $^1/_2$ as indicated in fig. 10.31.

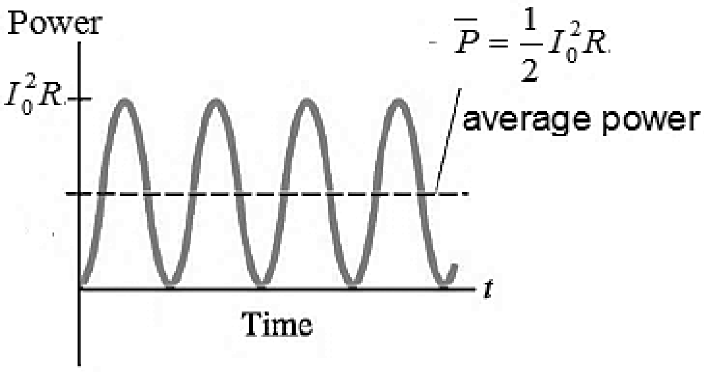


Fig. 10.31. Power delivered to a resistor in an AC circuit

Thus, the average power developed, is

$$\boldsymbol{\overline{P} = \frac{1}{2}I_0^2R.} \qquad (10.50)$$

Since power can also be written $\boldsymbol{P = U^2/R = (U_0^2\ R)\sin^2\omega t}$, we also have that the average power is

$$\boldsymbol{\overline{P} = \frac{1}{2}\frac{U_0^2}{R}.} \qquad (10.51)$$

The average or mean value of the square of the current or voltage is thus what is important for calculating average power: $\boldsymbol{\overline{I^2} = \frac{1}{2}I_0^2}$ and $\boldsymbol{\overline{U^2} = \frac{1}{2}U_0^2}$.

The square root of each of these is the ***rms*** (***root-mean-square***) value of the current or voltage:

$$\boldsymbol{I_{rms} = \sqrt{\overline{I^2}} = \frac{I_0}{\sqrt{2}} = 0.707I_0,} \qquad (10.52)$$

$$\boldsymbol{U_{rms} = \sqrt{\overline{U^2}} = \frac{U_0}{\sqrt{2}} = 0.707U_0}. \qquad (10.53)$$

The ***rms*** values of ***U*** and ***I*** are sometimes called the ***effective values***. They are useful because they can be substituted directly into the power formulas, equations (10.50, 10.51), to get the average power:

$$\boldsymbol{\overline{P} = \frac{1}{2} I_0^2 R = I_{rms}^2 R}; \qquad (10.54)$$

$$\boldsymbol{\overline{P} = \frac{1}{2}\frac{U_0^2}{R} = \frac{U_{rms}^2}{R}}. \qquad (10.55)$$

Thus, a direct current whose values of ***I*** and ***U*** equal the ***rms*** values of ***I*** and ***U*** for an alternating current will produce the same power. Hence it is usually the ***rms*** value of current and voltage that is specified or measured. For example, in a country, standard line voltage is 120 V ac. The 120 V is $\boldsymbol{U_{rms}}$; but the peak voltage $\boldsymbol{U_0}$ is

$$\boldsymbol{U_0 = \sqrt{2}U_{rms} = 170\ \text{V}}.$$

In much of the world (Europe, Australia, Asia) the ***rms*** voltage is 240 V, so the peak voltage is 340 V.

***Example 10.25*. Hair dryer.**

(a) Calculate the resistance and the peak current in a 1000-W hair dryer connected to a 120-V line; (b) What happens if it is connected to a 240-V line in Britain?

Solution. We are given ***P*** and $\boldsymbol{U_{rms}}$, so $I_{rms} = P/U_{rms}$, and $I_0 = \sqrt{2}I_{rms}$. Then we find ***R*** from $U = IR$.

a) We solve $\overline{P} = I_{rns}U_{rms}$ for the ***rms*** current:

$$I_{rms} = \frac{\overline{P}}{U_{rms}} = \frac{1000\ \text{W}}{120\ \text{V}} = 8.33\ \text{A}.$$

Then $I_0 = \sqrt{2}I_{rms} = 11.8$ A.

The resistance is $R = \dfrac{U_{rms}}{I_{rms}} = \dfrac{120\ \text{V}}{8.33\ \text{A}} = 14.4\ \Omega$.

b) When connected to a 240 V line, more current would flow and the resistance would change with the increased temperature. But let us make an estimate of the power transformed based on the same 14.4 Ω resistance. The average power would be $\overline{P} = \dfrac{U_{rms}^2}{R} = \dfrac{(240\ \text{V})^2}{(14.4\ \Omega)} = 4000$ W.

NOTE. *This is four times the dryer's power rating and would undoubtedly melt the heating element or the wire coils of the motor*.

Problems

1. A potential difference of 400 volt is applied across the ends of a conductor of resistance 80. Calculate the number of electrons flowing through it in one second. Charge of the electron is $1.6 \cdot 10^{-19}$ C. (Answer: $3.125 \cdot 10^{19}$)

2. A wire of resistance 1 Ω is drawn out so that its length is increased by twice its original length. Find its new resistance. (Answer: 4 Ω)

3. At room temperature (27.0 °C) the resistance of a heating element is 100 Ω. What is the temperature of the element if the resistance is found to be 117 Ω, given that the temperature coefficient of the material of the resistor is $1.70 \cdot 10^{-4}$ °C^{-1}. (Answer: 1027 °C)

4. Three resistors 2 Ω, 4 Ω and 5 Ω are combined in parallel. What is the total resistance of the combination? If the combination is connected to a battery of ***emf*** 20 V and negligible internal resistance, determine the current through each resistor, and the total current drawn from the battery. (Answer: 20/19 Ω, 10 A, 5 A, 4 A, 19 A)

5. A storage battery of ***emf*** 8.0 V and internal resistance of 0.5 Ω is being charged by a 120 V DC supply using a series resistor of 15.5 Ω. What is the terminal voltage of the battery during charging? What is the purpose of having a series resistor in the charging circuit? (Answer: 11.5 V)

6. A bulb of 484 Ω is producing light when connected to 220 V main supply. What is the electric power of the bulb? (Answer: 100 W)

7. An electric heater having resistance equal to 5 Ω is connected to electric source. If it produces 180 J of heat in one second, find the potential difference across the electric heater. (Answer: 30 V)

8. An ***ac*** voltage, whose peak value is 180 V, is across a 210 Ω resistor. What is the value of the ***rms*** and peak currents in the resistor? (Answer: 127 V, 606 mA)

Tests

1. Electric current flows when:
a) there is some potential difference;
b) there is some resistances;
c) atoms arrange themselves;
d) all of the above.

2. Ohm's law deals with the ratio between:
a) current and potential difference;
b) capacity and charge;
c) potential and capacity;
d) induced potential and flux.

3. When the resistance of the conductor is increased then the current will:
a) increase; b) decrease;
c) remains the same; d) none of these.

4. If the current ***I*** flowing through conductor under an applied potential ***U***, then according to Ohm's law:
a) $U \sim I^2$; b) $U \sim 1/I$; c) $I \sim U$; d) none of these.

5. A wire has resistance ***R***. Another wire identical to the first but having twice the diameter, has a resistance of:
a) R/4; b) 4/R; c) R/2; d) 2R.

6. Resistance of a conductor depends on its:
a) length; b) volume;
c) area of cross-section; d) temperature.

7. The resistance of a coil with a current of 12 A at 120 V is:
a) 0.1 Ω; b) 10 Ω; c) 1440 Ω; d) 132 Ω.

8. When the temperature of a metallic conductor is increased its resistance:
a) always decreases; b) always increases;
c) remains the same; d) None of these.

9. The reciprocal of resistance is:
a) specific resistance; b) effective resistance;
c) conductance; d) current.

10. Net resistance joined in series is:
a) sum of reciprocal of individual resistance;
b) sum of all the individual resistance;
c) product of resistances;
d) reciprocal of product of resistances.

11. The length of the conductor is halved. Then its conductivity be;
a) halved; b) doubled; c) unchanged; d) quadrupled.

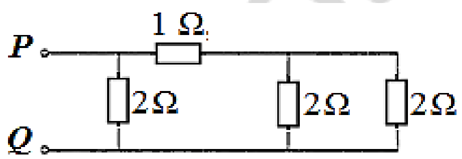


Fig. 10.32. Test 12

12. What is the equivalent resistance between points ***P*** and ***Q*** in the circuit shown in fig. 10.32?
a) 1 Ω; b) 2 Ω;
c) 3Ω; d) None of these.

13. The temperature coefficient of the resistance of a wire is 0.00125 $°C^{-1}$. At 300 °K its resistance is 1 Ω. The resistance will become 2 Ω at a temperature:
a) 1154 K; b) 1100 K; c) 1127 K; d) 1400 K.

14. There are three resistance 2, 3, 5 Ω connected in parallel to a battery of 10 V of negligible resistance. The potential drop across 3 Ω resistance is:
a) 2 V; b) 5 V; c) 3V; d) 10 V.

15. The electron (e = $1.6 \cdot 10^{-19}$ C) in hydrogen atom circles round the proton with a speed of $2.18 \cdot 10^{6}$ m/s in an orbit radius $5.3 \cdot 10^{-11}$ m. What is current constituted by it?

a) 1.05 mA; b) 10.5 mA;
c) 0.105 mA; d) none of the above.

16. Assuming that e = $1.6 \cdot 10^{-19}$ C, the number of electron passing per sec. through a wire carrying 1 A of current is:

a) $0.625 \cdot 10^{19}$; b) $1.5 \cdot 10^{19}$; c) $1.6 \cdot 10^{19}$; d) $6.25 \cdot 10^{16}$.

17. An electric heater operating at 220 V boils 5 liter of water in 5 minutes. If it is used on 110 V, it will boil the same amount of water in:

a) 10 minutes; b) 20 minutes;
c) 15 minutes; d) 25 minutes.

18. A constant voltage is applied between the two ends of a uniform metallic wire. Some heat is developed in it. The heat developed is doubled if:

a) the length of the wire is doubled;
b) the radius of the wire is doubled;
c) both the length and radius of the wire are doubled;
d) both the length and radius of the wire are halved.

19. A coil has a resistance 20 Ω at 0 °C and 21 Ω at 200 °C. What is the temperature coefficient of metal used for the coil:

a) $2 \cdot 10^{-4}$ K^{-1}; b) $2 \cdot 10^{-4}$ K^{-1}; c) $8 \cdot 10^{-4}$ K^{-1}; d) $1 \cdot 10^{-4}$ K^{-1}.

20. An electric lamp is marked 100 W. It is working on 200 V. The current through the lamp is given as:

a) 5A; b) 2 A; c) 0.5 A; d) 1.0 A.

21. Two electric bulbs whose resistances are in the ratio 1 : 2 are connected in series to a constant voltage source. The powers dissipated in them have the ratio:

a) 1 : 2; b) 1 : 1; c) 2 : 1; d) 4 : 1.

22. Two electric bulbs whose resistances are in the ratio 1 : 2 are connected in parallel to a constant voltage source. The powers dissipated in them have the ratio:

a) 1 : 2; b) 1 : 1; c) 2 : 1; d) 4 : 1.

23. ***Efm*** is measured in

a) Joule; b) Joule · Coulombs;
c) Joule / Coulomb; d) Joule / Coulomb/metre.

24. The *emf* of a battery is 3V and internal resistance 0.2 Ω. The difference of potential at the terminals of battery when connected across the external resistance of 1 Ω is:

a) 1.67 V; b) 2.5 V; c) 2.67 V; d) 3.67 V.

25. What is the peak current in a 2.8-kΩ resistor connected to a 120 V *rms* ac source.

a) 60.6 mA; b) 60 A; c) 6 A; d) 606 mA.

26. The peak value of an alternating current passing through a 1500 W device is 4.0 A. What is the *rms* voltage across it?

a) 530 mV; b) 530 V; c) 50 V; d) 350 V.

27. A heater coil connected to a 240 V ac line has a resistance of 40 Ω. What is the average power used?

a) 1440 W; b)140 W; c) 1000 W; d) 1400 W.

11. MAGNETIC FIELD

As well known all charges create electric fields, and these fields can be detected by other charges resulting in electric force. However, when charges are moved they create a new completely different field. This is the magnetic field. So any moving charge or an electric current produces a magnetic field in the surrounding space. In turn the magnetic field exerts a force only on moving charge or electric current and on the magnetized bodies.

The magnetic field is characterized by its magnitude $\boldsymbol{B}$ and direction. The magnetic field $\boldsymbol{B}$ is a vector and is represented by lines called lines of magnetic field. The vector $\boldsymbol{B}$ direction is tangent to this line at any point of the field. The strength of the magnetic field is proportional to the closeness of the lines.

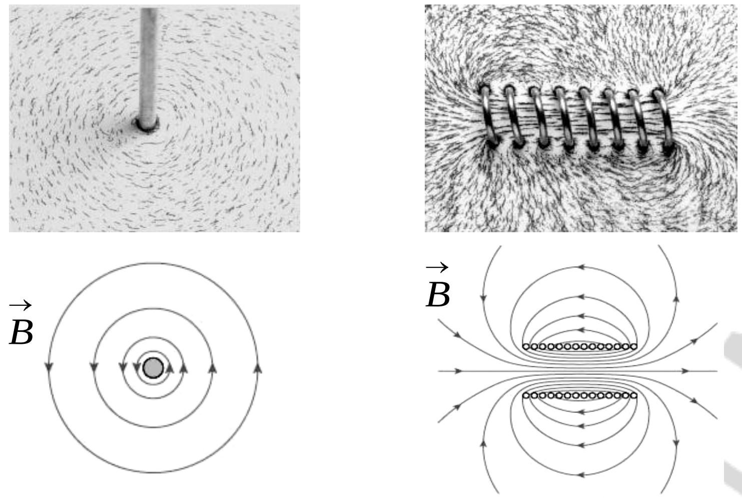
11.1. THE MAGNETIC FIELD PRODUCED BY ELECTRIC CURRENT

At first let's look a magnetic fields created by two main form of electric current: a magnetic fields of a straight current and a of current-carrying coil (solenoid) that illustrated in fig. 11.1.

A straight wire with electric current $\boldsymbol{I}$ going through it produces a magnetic field going in circles around it (fig. 11.1, *a*). The magnitude $\boldsymbol{B}$ of this magnetic field is equal to

$$B_A = \frac{\mu\mu_0 I}{2\pi b}, \tag{11.1}$$

where $\boldsymbol{I}$ is a value of electric current, $\boldsymbol{b}$ is a distance from the wire, $\boldsymbol{\mu}$ is the magnetic permeability of medium and $\boldsymbol{\mu_0 = 4\pi\cdot10^{-7}}$ **H/m** is magnetic constant (the magnetic permeability of vacuum).



a ***b***

Fig. 11.1. Magnetic fields $\vec{\boldsymbol{B}}$ of a straight current-carrying wire (*a*) and a current-carrying coil (solenoid) (*b*)

A solenoid is many loops of wire with a current going through. When the length of the solenoid is much larger than its radius, the magnetic field inside a solenoid is strong and uniform and is weak outside. The field lines inside the solenoid are nearly parallel, uniformly spaced, and close together (fig. 11.1, b).

The magnitude $\boldsymbol{B}$ of the magnetic field in a solenoid is equal to

$$\boldsymbol{B} = \frac{\mu\mu_0 \boldsymbol{I}}{2\boldsymbol{r}}\boldsymbol{n}, \tag{11.2}$$

where $\boldsymbol{I}$ is a value of electric current, $\boldsymbol{r}$ is a radius of a loop, $\boldsymbol{n}$ is an amount of the loops.

Magnetic field direction can be determined using the right hand rule (fig. 11.2). The right-hand rule gives the direction of the magnetic field lines that encircle a current-carrying wire. If the wire is grasped in the right hand with the thumb in the direction of the current, the fingers will curl in the direction of $\boldsymbol{B}$.

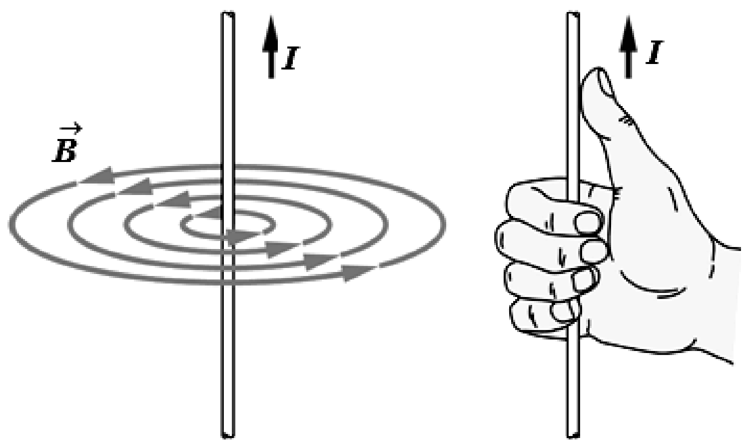


Fig. 11.2. Right-hand rule for determining the direction of $\vec{\boldsymbol{B}}$

11.2. FORCE ON AN ELECTRIC CURRENT IN A MAGNETIC FIELD (AMPERE'S FORCE)

It was found that a magnetic field exerts a force on a current-carrying wire.

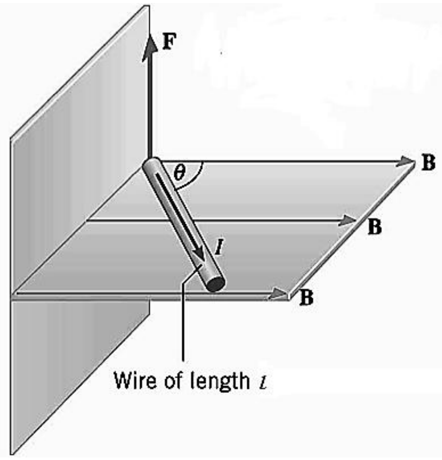


Fig. 11.3. Current-carrying wire of length l in a magnetic field

The force F (it is named after **Ampere**) on a wire carrying a current I with length l in a uniform magnetic field B is given by formula:

$$F = I \cdot l \cdot B \sin \theta. \qquad (11.3)$$

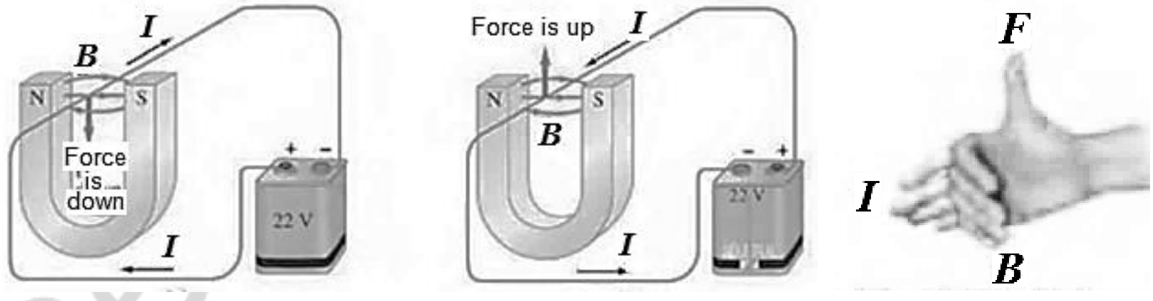
where **θ** is the angle between the current direction and the magnetic field B.

Equation (11.3) serves as a practical definition of B: $B = \dfrac{F}{I \cdot l}$, if $\theta = \pi/2$.

The SI unit for magnetic field B is the **tesla** (**T**), 1 **T** = 1 N/A·m. Another unit sometimes used to specify magnetic field is the **gauss** (G): 1 G = 10^{-4} T. It is necessary to note that the magnetic field of the Earth at its surface is about 0.5 G = 50 μT.

The direction of the force F is given by another right-hand rule, as illustrated in fig. 11.4, *c*. Orient your right hand so that outstretched fingers can point in the direction of the conventional current I, and when you bend your fingers they point in the direction of the magnetic field lines B. Then your thumb points in the direction of the force F on the wire.

As one can see from equation (11.3), the force F depends on the angle **θ** between the current direction and the magnetic field. When the current is perpendicular to the field lines ($\theta = 90°$) the force is strongest $F_{max} = IBl$. When the wire is parallel to the magnetic field lines ($\theta = 0°$) there is no force at all ($F = 0$). At other angles, the force F is proportional to $\sin\theta$.



a ***b*** ***c***

Fig. 11.4. (*a*) Force on a current-carrying wire placed in a magnetic field B; (*b*) same, but current I reversed; (*c*) right-hand rule for setup in (*b*)

***Example 11.1*. Magnetic force on a current-carrying wire.**

A wire carrying a 30 A current has a length l = 12 cm between the pole faces of a magnet at an angle θ = 60° (fig. 11.5). The magnetic field is approximately uniform at 0.90 T. We ignore the field beyond the pole pieces. What is the magnitude of the force on the wire?

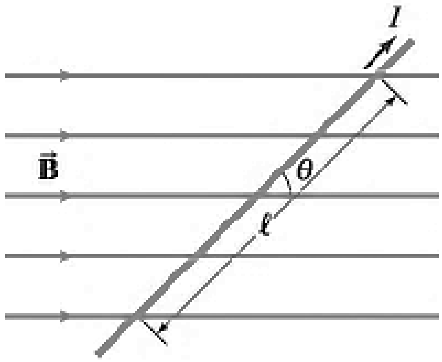


Fig. 11.5. Example 11.1. Current-carrying wire in a magnetic field. Force on the wire is directed into the page

Solution. The force ***F*** on the 12-cm length of wire within the uniform field ***B*** is $F = IlB\sin\theta$ = (30 A) · (0.12 m) · (0.90 T) · (0.866) = 2.8 N.

***Example 11.2*. Measuring a magnetic field.**

A rectangular loop of wire hangs vertically as shown in fig. 11.6. A magnetic field ***B*** is directed horizontally, perpendicular to the wire, and points out of the page at all points as represented by the symbol ⊙. The magnetic field ***B*** is very nearly uniform along the horizontal portion of wire **ab** (length ***l*** = 10.0 cm) which is near the center of the gap of a large magnet producing the field. The top portion of the wire loop is free of the field. The loop hangs from a balance which measures a downward magnetic force (in addition to the gravitational force) of ***F*** = $3.48\cdot10^{-2}$ *N* when the wire carries a current ***I*** = 0.245 A. What is the magnetic field at the center of magnet?

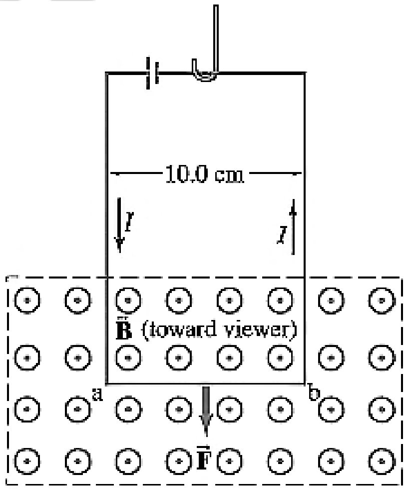


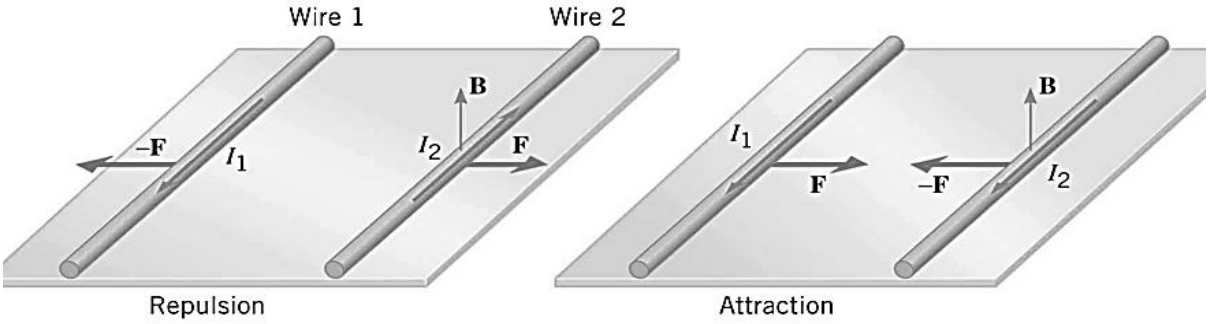
Fig. 11.6. Measuring a magnetic field ***B***

Solution. Three straight sections of the wire loop are in the magnetic field: a horizontal section and two vertical sections. We apply $F = IlB\sin\theta$ to each section and use the right-hand rule. The magnetic force on the left vertical section of wire points to the left; the force on the vertical section on the right points to the right. These two forces are equal and in opposite directions and so add up to zero. Hence, the net magnetic force on the loop is that on the horizontal section **ab**, whose length is ***l*** = 0.100 m. The angle **θ** between ***B*** and the wire is θ = 90°, so sinθ = 1. Thus

$$B = \frac{F}{I\cdot l} = \frac{3.48\cdot10^{-2}\,N}{(0.245\text{ A})\cdot(0.100\text{ m})} = 1.42\text{ T.}$$

NOTE. *This technique can be a precise means of determining magnetic field strength.*

Since a current in a wire creates its own magnetic field, two current carrying wires placed close together exert magnetic forces on each other. Parallel wires with current flowing in the same direction, attract each other and parallel wires with current flowing in the opposite direction, repel each other (fig. 11.7).



a ***b***

Fig. 11.7. Antiparallel currents (in opposite direction) exert a repulsive force on each other (*a*). Parallel currents in the same direction exert an attractive force on each other (*b*)

The force between two parallel wires carrying a current is used to define the SI unit of current is Ampere: one **Ampere** is definite as that current flowing in each of two long parallel conductors 1m apart, which result in a force of exactly $2 \cdot 10^{-7}$ N/m of 1 m length of each conductor.

11.3. FORCE ON AN ELECTRIC CHARGE MOVING IN A MAGNETIC FIELD (LORENTZ'S FORCE)

If a particles of charge $\boldsymbol{q}$ moves through a magnetic field $\boldsymbol{B}$ with a velocity $\boldsymbol{\upsilon}$ then a force $\boldsymbol{F}$ (named after **Lorentz**) acts on it:

$$\boldsymbol{F} = \boldsymbol{q\upsilon B}\mathbf{sin\theta}, \qquad (11.4)$$

$\boldsymbol{\theta}$ is angle between $\boldsymbol{\upsilon}$ and $\boldsymbol{B}$. The force $\boldsymbol{F}$ is greatest when the particle moves perpendicular to $\boldsymbol{B}$ ($\boldsymbol{\theta} = 90°$): $\boldsymbol{F}_{\mathbf{max}} = \boldsymbol{q\upsilon B}$.

The force $\boldsymbol{F}$ is zero, if the particle moves parallel to the field lines $\boldsymbol{B}$ ($\boldsymbol{\theta} = \mathbf{0°}$). The direction of the force $\boldsymbol{F}$ is perpendicular to the magnetic field $\boldsymbol{B}$ and to the velocity $\boldsymbol{\upsilon}$ of the particle. It is given again by a right-hand rule (***for*** $\boldsymbol{q} > \mathbf{0}$): you orient your right hand so that your outstretched fingers point along the direction of the particle's velocity $\boldsymbol{\upsilon}$ and when you bend your fingers they must point along the direction of $\boldsymbol{B}$. Then your thumb will point in the direction of the force $\boldsymbol{F}$. This is true only for positively charged particles, and will be "up" for the positive particle shown in fig. 11.8. For negatively charged particles, the force is in exactly the opposite direction "down" in fig. 11.8.

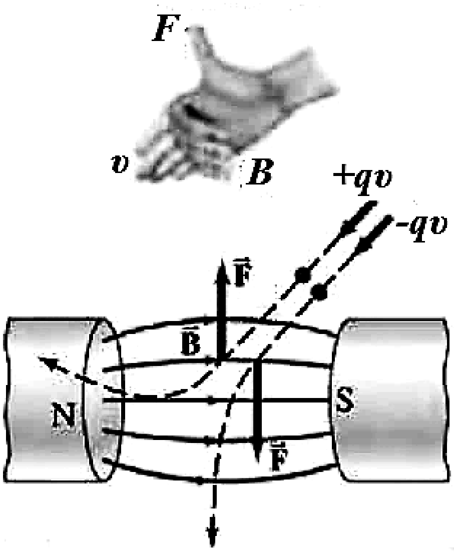


Fig. 11.8. Force on charged particles due to a magnetic field is perpendicular to the magnetic field direction. If ***υ*** is horizontal, then ***F*** is vertical

***Example 11.3*. Magnetic force on ions during a nerve pulse**.

Estimate the magnetic force due to the Earth's magnetic field on ions crossing a cell membrane during an action potential. Assume the speed of the ions is 10^{-2} m/s.

Solution. Using $F = q\ \upsilon\ B$, set the magnetic field of the Earth to be roughly $B \approx 10^{-4}$ T, and the charge $q \approx e \approx 10^{-19}$ C.

$$F \approx (10^{-19}\ \text{C}) \cdot (10^{-2}\ \text{m/s}) \cdot (10^{-4}\ \text{T}) = 10^{-25}\ \text{N}.$$

NOTE. *This is an extremely small force. Yet it is thought migrating animals do somehow detect the Earth's magnetic field, and this is an area of active research.*

***Example 11.4*. Electron's path in a uniform magnetic field.**

An electron travels at $2.0 \cdot 10^7$ m/s in a plane perpendicular to a uniform magnetic field of 10 mT. Describe its path quantitatively.

Solution. An electron at point **P** (fig. 11.9) is moving to the right, and the force on it at this point is downward as shown (use the right-hand rule and reverse the direction for negative charge). The electron is thus deflected toward the page bottom. A moment later, say, when it reaches point **Q**, the force is still perpendicular to the velocity and is in the direction shown.

Because the force is always perpendicular to $\boldsymbol{\upsilon}$, the magnitude of $\boldsymbol{\upsilon}$ does not change — the electron moves at constant speed. If the force on a particle is always perpendicular to its velocity $\boldsymbol{\upsilon}$, the particle moves in a circle (fig. 11.9) and has a centripetal acceleration $\boldsymbol{a} = \boldsymbol{\upsilon}^2/\boldsymbol{r}$.

We find the radius of curvature using Newton's second law. The force is given by $\boldsymbol{F} = \boldsymbol{q}\ \boldsymbol{\upsilon}\ \boldsymbol{B}$ as $\sin\theta = 1$.

$$F = ma.$$

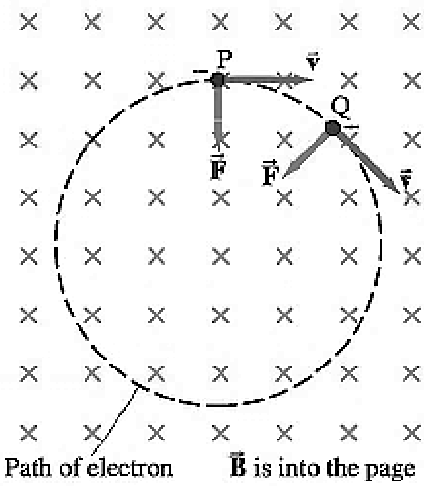


Fig. 11.9. Example 11.4. Force ***F*** exerted by a uniform magnetic field on a moving charged particle (in this case, an electron) produces a circular path

We insert ***F*** and ***a*** into Newton's second law: $q\upsilon B = \dfrac{m\upsilon^2}{r}$.

We solve it for ***r*** and find the radius of circle and period ***T*** of rotation

$$r = \frac{m\upsilon}{qB}, \qquad T = \frac{2\pi r}{\upsilon} = \frac{2\pi m}{qB}.$$

The radius of circle is directly proportional to speed of a charge, period ***T*** of a charge rotation doesn't depend on a speed magnitude and both values are inverse proportional to magnitude of magnetic field. To get ***r*** we put in the numbers:

$$r = \frac{(9.1 \cdot 10^{-31}\,\text{kg}) \cdot (2.9 \cdot 10^{7}\,\text{m/s})}{(1.6 \cdot 10^{-19}\,\text{C}) \cdot (0.010\ \text{T})} = 1.1 \cdot 10^{-2}\,\text{m} = 1.1\ \text{cm}.$$

NOTE. *Thus a charged particle moves in a circular path with constant centripetal acceleration in a uniform magnetic field.*

PROBLEMS

1. a) What is the force per meter of length on a straight wire carrying a 7.40 A current when perpendicular to a 0.90 T uniform magnetic field? b) What if the angle between the wire and field is 45.0°? (Answer: a) 6.7 N/m; b) 4.7 N/m)

2. How much current is flowing in a wire 4.20 m long if the maximum force on it is 0.900 N when placed in a uniform 0.0800 T field? (Answer: 2.68 A)

3. In a magnetic field with B = 0.5 T, for what path radius, will an electron circulate at 0.1 the speed of light ($c = 3 \cdot 10^{8}$ m/s)? What will be its kinetic energy? Given mass of electron $9.1 \cdot 10^{-31}$ kg, its charge $-1.6 \cdot 10^{-19}$ C. (Answer: $3.41 \cdot 10^{-4}$ m; $4.095 \cdot 10^{-16}$ J)

TESTS

1. SI unit of the magnetic field ***B*** is:

a) Oersted; b) Tesla; c) Gauss; d) Maxwell.

2. Current is passed through a straight wire. The lines of magnetic field produced by it are:

a) circular and endless;
b) oval in shape but endless;
c) straight but endless;
d) None of these.

3. In an uniform magnetic field, lines of magnetic field are:

a) inclined; b) parallel; c) circular: d) perpendicular.

4. When the direction of current is opposite in two parallel wires placed near each other, they will:

a) attract each other; b) repel each other;
c) neither attract nor repel; d) sometimes attract sometimes repel.

5. Force acting on a moving charge in a magnetic field varies with the velocity υ as:

a) $1/\upsilon$; b) υ; c) υ^2; d) $\upsilon^{½}$.

6. Force acting on a moving charge in a magnetic field is maximum when velocity of charge is inclined to field at:

a) 0°; b) 60°; c) 30°; d) 90°.

7. A wire of length 2 m carries a current of 10 A. What is the force acting on it when it is placed at an angle 45° to the uniform magnetic field of 0.15 T:

a) 1.5 N; b) 3N; c) $3\sqrt{2}N$; d) $\frac{3}{\sqrt{2}}N$.

8. An electron is moving parallel to the magnetic field ***B*** with velocity **υ**. The force acting on it is:

a) Beυ; b) Be/υ; c) zero; d) eυ/B.

9. The force acting on a charge q moving with velocity υ in the magnetic field *B* is given by:

a) q/ υB; b) υB/q; c) qυB; d) υ/Bq.

10. An electron moving with velocity 10^6 ms^{-1} enters a magnetic field and describes a circle of radius 0.1 m, the magnetic field ***B*** is:

a) $1.8\cdot10^{-4}$ T; b) $5.5\cdot10^{-5}$ T; c) $1.4\cdot10^{-5}$ T; d) $14\cdot10^{-5}$ T.

11. An electron is moving with velocity $3\cdot10^7$ ms^{-1} perpendicular to the magnetic field of 2.0 T. The magnitude of the force acting on it is:

a) $96\cdot10^{-12}$ N; b) $9.6\cdot10^{-12}$ N;
c) $0.96\cdot10^{-12}$ N; d) none of the above.

12. A particle having charge ***q*** enters with a uniform velocity **υ** in the magnetic field ***B***, the radius of the path in which it moves is (***m*** is the mass of particle):

a) *m*υ/Bq; b) Bυ/mq; c) mB/qυ; d) none of the above.

11.4. ELECTROMAGNETIC INDUCTION AND FARADAY'S LAW

We know that there are two ways in which electricity and magnetism are related:

1) an electric current produces a magnetic field;

2) a magnetic field exerts a force on an electric current or moving electric charge.

Here we deal with the reverse phenomena i. e. the production of an electric current from a magnetic field. This phenomenon of producing an electric current from a magnet or a magnetic field is known as electromagnetic induction.

Faraday has found that a steady magnetic field produces no current in a conductor (fig. 11.10, *c*), but a changing magnetic field can produce an electric current (Fig. 11.10, *a*, *b*). Such a current is called an induced current. When the magnetic field changes, a current flows as if there were a source of ***emf*** in the circuit. Therefore an induced ***emf*** is produced by a changing magnetic field.

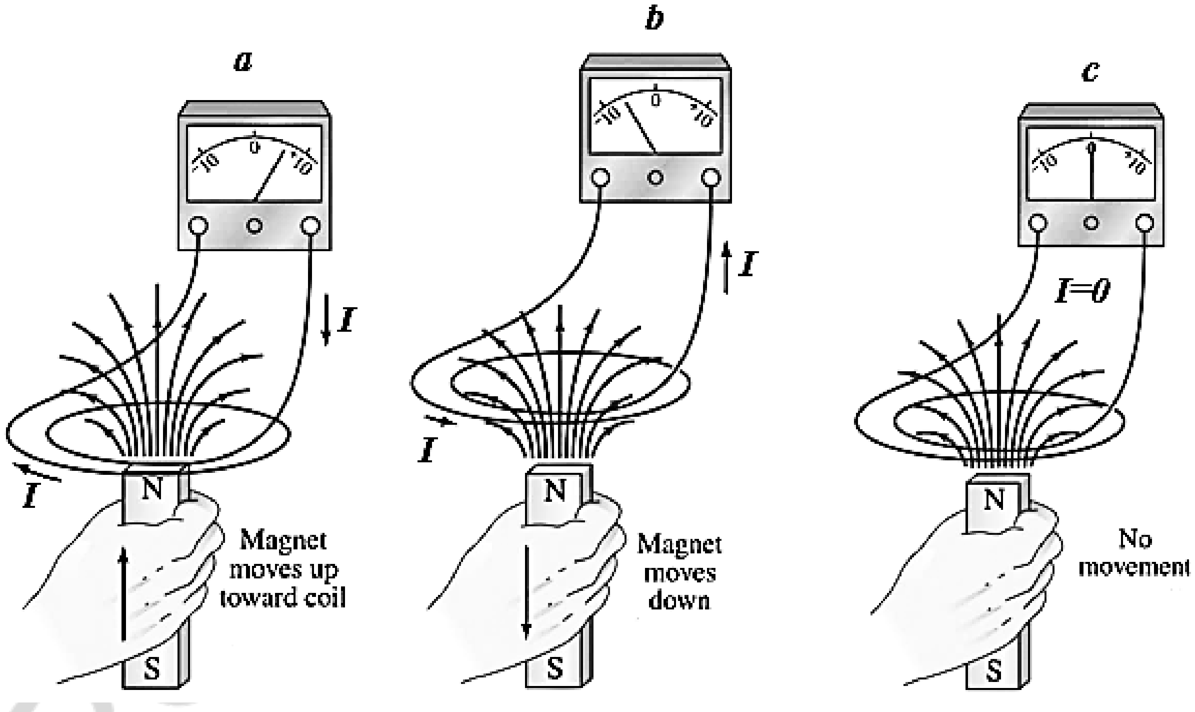


Fig. 11.10. (*a*) A current is induced when a magnet is moved toward a coil. (*b*) The induced current is opposite when the magnet is moved away from the coil. Note that the galvanometer zero is at the center of the scale and the needle deflects left or right, depending on the direction of the current. In (*c*) no current is induced if the magnet does not move relative to the coil. It is the relative motion that counts: the magnet can be held steady and the coil moved, which also induces an ***emf***

Fig. 11.10 shows that if a magnet is moved quickly into a coil of wire, a current is induced in the wire. If the magnet is quickly removed, a current is induced in the opposite direction (***B*** through the coil decreases). Furthermore, if the magnet is held steady and the coil of wire is moved toward or away from the magnet, again an ***emf*** is induced and a current flows. Motion or change is required to induce an ***emf***. It doesn't matter whether the magnet or the coil moves. It is their relative motion that counts.

11.5. FARADAY'S LAW OF INDUCTION

Faraday investigated quantitatively what factors influence the magnitude of the ***emf*** induced. He found first of all that the more rapidly the magnetic field changes, the greater the induced ***emf***. But the ***emf*** is not simply proportional to the rate of change of the magnetic field, ***B***. Rather the ***emf*** is proportional to the rate of change of the magnetic flux, $\boldsymbol{\Phi}$, passing through the circuit or loop of area ***S***. Magnetic flux $\boldsymbol{\Phi}$ for a uniform magnetic field is defined as

$$\boldsymbol{\Phi} = \boldsymbol{B}_{\perp}\boldsymbol{S} = \boldsymbol{BS}\cos\boldsymbol{\alpha}, \tag{11.5}$$

here $\boldsymbol{B}_{\perp}$ is the component of the magnetic field ***B*** perpendicular to the face of the loop, and $\boldsymbol{\alpha}$ is the angle between ***B*** and normal to the surface of the loop (fig. 11.11).

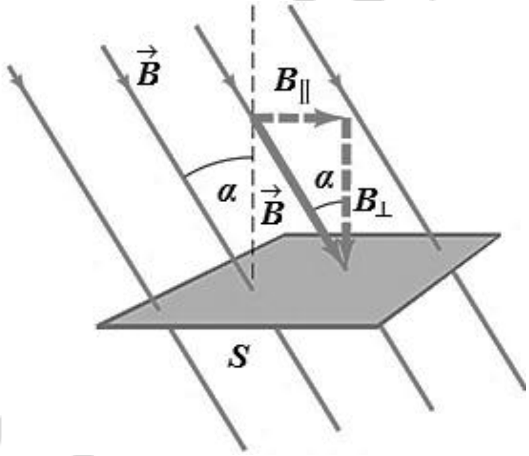


Fig. 11.11. Determining the magnetic flux through a flat loop of wire of area ***S***

The unit of magnetic flux $\boldsymbol{\Phi}$ is called a weber: 1 Wb = 1 T·m^2. Hence, one Weber may be defined as the amount of magnetic flux produced by uniform magnetic field of 1 Tesla normal to an area of 1 meter^2.

With this definition of flux $\boldsymbol{\Phi}$ one can write down the results of Faraday's investigations: the ***emf*** $\boldsymbol{\varepsilon_i}$ induced in a circuit is equal to the rate of change of magnetic flux through the circuit:

$$\varepsilon_i = -\frac{\Delta\boldsymbol{\Phi}}{\Delta\boldsymbol{t}}. \tag{11.6}$$

This fundamental result is known as ***Faraday's law of induction***, and is one of the basic laws of electromagnetism.

If the circuit contains $\boldsymbol{N}$ loops that are closely wrapped so the same flux passes through each, the ***emfs*** induced in each loop add together, so

$$\boldsymbol{\varepsilon_i = -N\frac{\Delta\Phi}{\Delta t}}. \qquad (11.7)$$

Example 11.5. **A loop of wire in a magnetic field.**

A square loop of wire of side $\boldsymbol{l}$ = 5.0 cm is in a uniform magnetic field $\boldsymbol{B}$ = 0.16 T. What is the magnetic flux in the loop (a) when $\boldsymbol{B}$ is perpendicular to the face of the loop and (b) when $\boldsymbol{B}$ is at an angle of 30° to the area $\boldsymbol{S}$ of the loop? (c) What is the magnitude of the average current in the loop if it has a resistance of 0.012 Ω and it is rotated from position (b) to position (a) in 0.14 s?

Solution. We use the definition $\Phi_B = BS$ to calculate the magnetic flux. Then we use Faraday's law of induction to find the induced ***emf*** in the coil, and from that the induced current ($I = \varepsilon/R$).

The area of the coil is $\boldsymbol{S = l^2 =} (5.0\cdot10^{-2}\text{ m})^2 = 2.5\cdot10^{-3}\text{ m}^2$, and the direction of $\boldsymbol{S}$ is perpendicular to the face of the loop.

a) $\boldsymbol{B}$ is perpendicular to the coil's face, and thus parallel to $\boldsymbol{S}$, so

$$\Phi_B = BS\cos0^\circ = (0.16\text{ T})\cdot(2.5\cdot10^{-3}\text{ m}^2)\cdot l = 4.0\cdot10^{-4}\text{ Wb}.$$

b) The angle between $\boldsymbol{B}$ and $\boldsymbol{S}$ is 30°, so

$$\Phi_B = BS\cos\theta^\circ = (0.16\text{ T})\cdot(2.5\cdot10^{-3}\text{ m}^2)\cdot\cos30^\circ = 3.5\cdot10^{-4}\text{ Wb}.$$

c) The magnitude of the induced *emf* is

$$I = \frac{\varepsilon}{R} = \frac{3.6\cdot10^{-4}\,\text{V}}{0.012\ \Omega} = 0.030\text{ A} = 30\text{ mA}.$$

The current is then $I = \dfrac{\varepsilon}{R} = \dfrac{3.6\cdot10^{-4}\,\text{V}}{0.012\ \Omega} = 0.030\text{ A} = 30\text{ mA}.$

The minus signs in equation (11.6) is placed there to remind us in which direction the induced ***emf*** $\boldsymbol{\varepsilon_i}$ acts. Experiments show that ***an induced emf*** $\boldsymbol{\varepsilon_i}$ ***gives rise to a current whose magnetic field opposes the original change in flux.*** This is known as ***Lenz's law***.

Let's apply Lenz's law to the relative motion between a magnet and a coil in fig. 11.10. The changing flux through the coil induces an ***emf*** $\boldsymbol{\varepsilon_i}$, which produces a current in the coil. And this induced current produces its own magnetic field. In fig. 11.10, *a* the distance between the coil and the magnet decreases. The magnetic field (and number of field lines), and therefore the flux through the coil increases. The magnetic field of the magnet points upward. To oppose this upward increase, the magnetic field inside the coil produced by the induced current points downward. Thus, Lenz's law tells us that the current moves as shown (use the right-hand rule). In fig. 11.10, *b* the flux decreases (because the magnet is moved away), so the induced current produces an upward magnetic field through the coil that is "trying" to maintain the status quo. Thus the current in fig. 11.10, *b* is in the opposite direction from fig. 11.10, *a*.

According to equation (11.5) induced ***emf*** ε_i can be produced by any one of the following methods:

by changing the magnetic field ***B***;

by changing the area ***S*** of the circuit;

by changing the angle α, i. e. the relative orientation of the field ***B*** and area ***S***.

Consider induced ***emf*** ε_i which is produced by changing the area ***S*** of the circuit (fig. 11.12). Assume that a uniform magnetic field ***B*** is perpendicular to the area bounded by the U-shaped conductor and the movable rod resting on it.

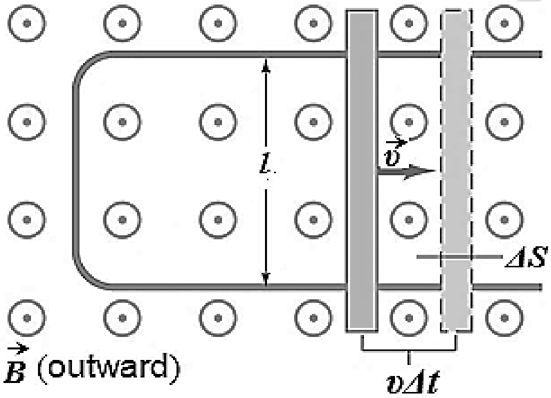


Fig. 11.12. A conducting rod is moved to the right on a U-shaped conductor in a uniform magnetic field ***B*** that points out of the page

If the rod moves at a speed υ, it travels a distance $\Delta x = \upsilon\Delta t$ in a time Δt. Therefore, the area of the loop increases by an amount $\Delta S = l\,\Delta x = l\,\upsilon\Delta t$ in a time Δt. By Faraday's law there is an induced ***emf*** ε_i whose magnitude is given by:

$$\varepsilon_i = \frac{\Delta \Phi}{\Delta t} = \frac{B\Delta S}{\Delta t} = \frac{Bl\upsilon\Delta t}{\Delta t} = Bl\upsilon. \qquad (11.8)$$

This equation (11.8) is valid as long as ***B***, ***l*** and υ are mutually perpendicular. An ***emf*** ε_i induced by a conductor moving with a velocity υ at an angle α with a magnetic field ***B*** is given by:

$$\varepsilon_i = Bl\upsilon\sin\alpha. \qquad (11.9)$$

An ***emf*** ε_i induced on a conductor moving in a magnetic field is sometimes called ***motional emf***.

Right hand rule is used to determine the direction of the induced ***efm*** ε_i of a conductor moving in a magnetic field (fig. 11.13). If the first finger points in the direction of the magnetic field, the thumb points in the direction of motion of the conductor, then the central finger points in the direction of the induced ***efm***.

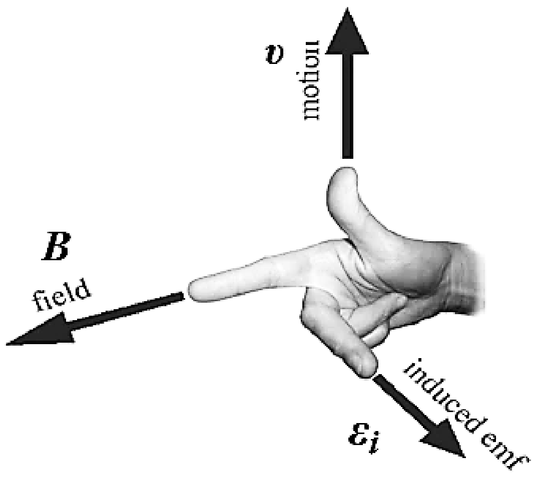


Fig. 11.13. Right hand rule for determining the direction of the induced ***efm*** ε_i of a conductor moving with velocity ***v*** in a magnetic field ***B***

***Example 11.6*. Pulling a coil from a magnetic field.**

A 100-loop square coil of wire, with side ***l*** = 5.00 cm and total resistance 100 Ω, is positioned perpendicular to a uniform 0.600 T magnetic field, as shown in fig. 11.14.

It is quickly pulled from the field at constant speed (moving perpendicular to ***B***) to a region where ***B*** drops abruptly to zero. At ***t*** = 0, the right edge of the coil is at the edge of the field. It takes 0.100 s for the whole coil to reach the field-free region. Find: a) the rate of change in flux through the coil; *b*) the ***emf*** and current induced; c) how much energy is dissipated in the coil? d) What was the average force required ($\boldsymbol{F}_{ext}$)?

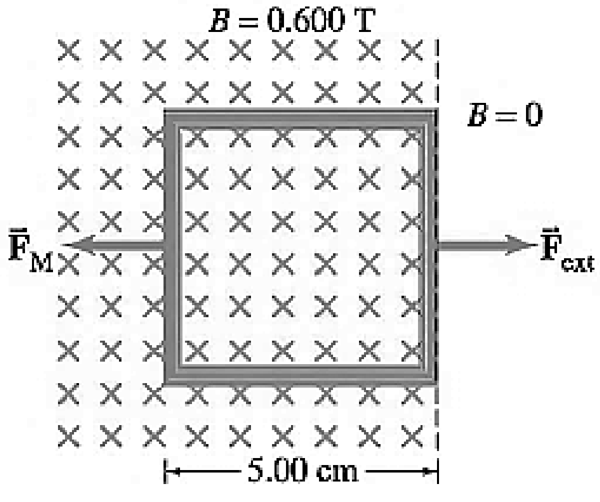


Fig. 11.14. Example 11.6. The square coil in a magnetic field ***B*** = 0.600 T is pulled abruptly to the right to a region where ***B*** = 0

Solution. We start by finding how the magnetic flux, $\Phi_B = BS$, changes during the time interval Δ***t*** = 0.100 s. Faraday's law then gives the induced ***emf*** and Ohm's law gives the current.

a) The area of the coil is $\boldsymbol{S} = \boldsymbol{l}^2$ = $(5.00\cdot10^{-2}\ \text{m})^2 = 2.50\cdot10^{-3}\ \text{m}^2$. The flux through one loop is initially $\Phi_B = BS = (0.600\ \text{T})\cdot(2.50\cdot10^{-3}\ \text{m}^2) = 1.50\cdot10^{-3}$ Wb. After 0.100 s, the flux is zero. The rate of change in flux is constant (because the coil is square), equal to $\dfrac{\Delta\Phi_B}{\Delta t} = \dfrac{0-(1.50\cdot10^{-3}\,\text{Wb})}{0.100\ \text{s}} = -1.50\cdot10^{-2}\,\text{Wb/s}.$

b) The ***emf*** induced in the 100-loop coil during this 0.100 s interval is

$$\varepsilon = -N\frac{\Delta\Phi_B}{\Delta t} = (-100)\cdot(-1.50\cdot10^{-2}\,\text{Wb/s}) = 1.50\ \text{V}.$$

The current is found by applying Ohm's law to the 100-Ω coil:

$$I = \frac{\varepsilon}{R} = \frac{1.50\ \text{V}}{100\ \Omega} = 1.50\cdot10^{-2}\,\text{A} = 15.0\ \text{mA}.$$

By Lenz's law, the current must be clockwise to produce more ***B*** into the page and thus oppose the decreasing flux into the page.

c) The total energy dissipated in the coil is the product of the power ($P = I^2R$) and the time ***t***:

$$E = Pt = I^2Rt = (1.50\cdot10^{-2}\ \text{A})^2\cdot(100\ \Omega)\cdot(0.100\ \text{s}) = 2.25\cdot10^{-3}\ \text{J}.$$

d) We can use the result of part (c) and apply the work-energy principle: the energy dissipated ***E*** is equal to the work ***A*** needed to pull the coil out of the field. Because $A = Fd$ where $\boldsymbol{d}$ = 5.00 cm, then

$$\vec{F} = \frac{A}{d} = \frac{2.25\cdot10^{-3}\,\text{J}}{5.00\cdot10^{-2}\,\text{m}} = 0.0450\ \text{N}.$$

Alternate Solution (d) We can also calculate the force directly using $F = I\cdot l\cdot B$, which here for constant ***B*** is $F = IlB$. The force the magnetic field exerts on the top and bottom sections of the square coil of fig. 11.14 are in opposite directions and cancel each other. The magnetic force $\boldsymbol{F_m}$ exerted on the left vertical section of the square coil acts to the left as shown because the current is up (clockwise). The right side of the loop is in the region where ***B*** = 0. Hence the external force, to the right, needed to just overcome the magnetic force to the left (on ***N*** = 100 loops) is

$$\boldsymbol{F_{ext}} = \boldsymbol{NIlB} = (100)\cdot(0.0150\ \text{A})\cdot(0.0500\ \text{m})\cdot(0.600\ \text{T}) = 0.0450\ \text{N},$$

which is the same answer, confirming our use of energy conservation above.

***Example 11.7*.** **Does a moving airplane develop a large *emf*?**

An airplane travels 1000 km/h in a region where the Earth's magnetic field is about $5\cdot10^{-5}$ T and is nearly vertical (fig. 11.15). What is the potential difference induced between the wing tips that are 70 m apart?

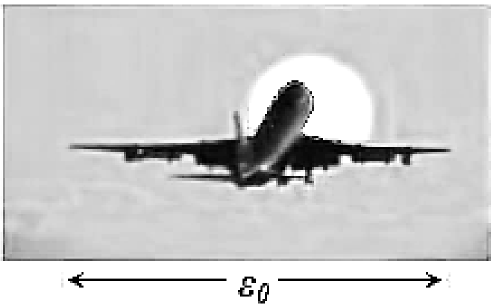


Fig. 11.15. Example 11.7. A moving airplane

Solution. We consider the wings to be a 70-m-long conductor moving through the Earth's magnetic field. We use equation (11.9) to get the ***emf***.

Since υ = 1000 km/h = 280 m/s, and $\vec{\upsilon} \perp \vec{B}$, we have $\varepsilon_i = Bl\upsilon$

$$\varepsilon_i = Bl\upsilon = (5\cdot10^{-5}\ \text{T})\cdot 70\ \text{m} \cdot 280\ \text{m/s} \approx 1\ \text{V}.$$

***Example 11.8*. Electromagnetic blood-flow measurement.**

The rate of blood flow in our body's vessels can be measured using the apparatus shown in fig. 11.16, since blood contains charged ions. Suppose that the blood vessel is 2.0 mm in diameter, the magnetic field is 0.080 T, and the measured ***emf*** is 0.10 mV. What is the flow velocity **υ** of the blood?

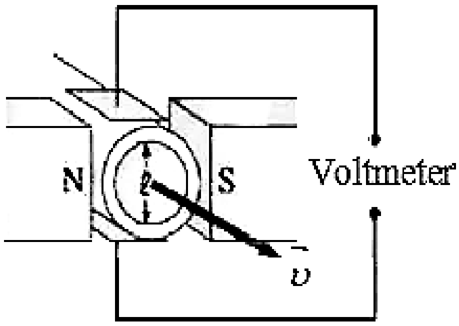


Fig. 11.16. Example 11.8. Measurement of blood velocity from the induced ***emf***

Solution. The magnetic field ***B*** points horizontally from left to right (***N*** pole toward ***S*** pole). The induced ***emf*** acts over the width ***l*** = 2.0 mm of the blood vessel, perpendicular to ***B*** and **υ** (fig. 11.16). We can then use equation $\varepsilon_i = Bl\upsilon$ to get **υ**. We solve for **υ** in equation $\varepsilon_i = Bl\upsilon$:

$$\upsilon = \frac{\varepsilon_i}{Bl} = \frac{(1.0\cdot10^{-4}\text{V})}{(0.080\ \text{T})\cdot(2.0\cdot10^{-3}\text{m})} = 0.63\ \text{m/s}.$$

11.6. SELF-INDUCTANCE

Let us consider circuit contains a coil of ***N*** turns (fig. 11.17). When a changing current passes through the coil (or solenoid), a changing magnetic flux is produced inside the coil, and this in turn induces an ***emf*** in that same coil (fig. 11.17). This induced ***emf*** opposes the change in flux (Lenz's law). For example, if the current through the coil is increasing, the increasing magnetic flux induces an ***emf*** that opposes the original current and tends to retard its increase. If the current is decreasing in the coil, the decreasing flux induces an ***emf*** in the same direction as the current, thus tending to maintain the original current.

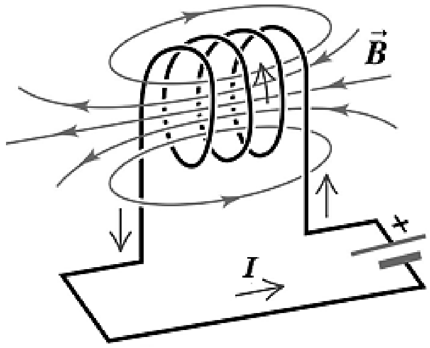


Fig. 11.17. The current ***I*** in the circuit causes a magnetic field ***B*** in the coil and hence a magnetic flux ***Φ*** through the coil. When the current changes, the flux ***Φ*** changes also and a self-induced *emf* appears

Self-inductance occurs when a changing current in a circuit results in an induced ***emf*** that opposes the change in the circuit itself. Self-inductance occurs because some of the magnetic flux produced in a circuit passes through that same circuit.

For a coil carrying current ***I***, there is a magnetic field produced around it. The value of ***B*** at each point is proportional to the current. Therefore the magnetic flux ***Φ*** passing through the every loop of the coil is also proportional to the current ***I*** in the coil:

$$\boldsymbol{\Phi} \sim \boldsymbol{I} \quad \text{or} \quad \boldsymbol{\Phi} = \boldsymbol{LI}, \tag{11.10}$$

where ***L*** is constant of proportionality and is called the ***self-inductance*** or ***inductance***.

If current ***I*** is equal to 1 (unit) then from equation (11.10) ***L = Φ***. Hence, the self-inductance ***L*** of a circuit is equal to the flux linked with it when a unit current flows through the circuit.

According to Faraday's law the ***emf*** ε_i induced in a coil of self-inductance ***L*** is:

$$\varepsilon_i = -N\frac{\Delta\Phi}{\Delta t} = -L\frac{\Delta I}{\Delta t}. \tag{11.11}$$

Let $\Delta I/\Delta t$ be equal to a unit then the ***emf*** $\varepsilon_i = -L$. Hence, self-inductance ***L*** of a circuit is numerically equal to the induced ***emf*** ε_i set up in it, when the rate of change of current through the circuit is unity. The self-inductance ***L*** is measured in henries ***H***:

$$1\ \text{Henry} = \frac{1\ \text{Volt}}{1\ \text{ampere/sec}}.$$

The magnitude of ***L*** depends on the geometry and on the presence of a ferromagnetic material.

Circuits always contain some inductance, but often it is quite small unless the circuit contains a coil of many turns. A coil that has significant self-inductance ***L*** is called an inductor.

Inductance is shown on circuit diagrams by the symbol: .

***Example 11.9*. Solenoid inductance.**

(a) Determine a formula for the self-inductance $\boldsymbol{L}$ of a tightly wrapped and long solenoid containing $\boldsymbol{N}$ turns of wire in its length $\boldsymbol{l}$ and whose cross-sectional area is $\boldsymbol{S}$. (b) Calculate the value of $\boldsymbol{L}$ if $\boldsymbol{N}$ = 100, $\boldsymbol{l}$ = 5.0 cm, $\boldsymbol{S}$ = 0.30 cm^2 and the solenoid is air filled.

Solution. To determine the inductance $\boldsymbol{L}$, it is usually simplest to start with equation $L = \dfrac{N\Phi}{I}$, so we need to first determine the flux $\boldsymbol{\Phi}$.

The magnetic field inside a solenoid (ignoring end effects) is constant: $\boldsymbol{B = \mu_o nI}$, where the number of loops per unit length $\boldsymbol{n = N/l}$.

The flux is $\Phi = BS = \mu_o NIS/l$, so $L = \dfrac{N\Phi}{I} = \dfrac{\mu_0 N^2 S}{l}$.

b) Since $\mu_o = 4\pi\cdot 10^{-7}$ T·m/A, then

$$L = \frac{(4\pi\cdot 10^{-7}\,\mathrm{T\cdot m/A})\cdot(100)^2\cdot(3.0\cdot 10^{-5}\,\mathrm{m}^2)}{5.0\cdot 10^{-2}\,\mathrm{m}} = 7.5\ \mu\mathrm{H}.$$

NOTE. *Magnetic field lines “stray” out of the solenoid (see fig. 11.1, b), especially near the ends, so our formula is only an approximation.*

***Example 11.10*.**

A solenoid that has a length equal to 25.0 cm, a radius equal to 0.800 cm, and 400 turns is in a region where a magnetic field of 600 G exists and makes an angle of 60° with the axis of the solenoid. (*a*) Find the magnetic flux through the solenoid. (*b*) Find the magnitude of the average ***emf*** induced in the solenoid if the magnetic field is reduced to zero in 1.40 s.

Solution. We can use its definition to find the magnetic flux through the solenoid and Faraday’s law to find the ***emf*** induced in the solenoid when the external field is reduced to zero in 1.4 s.

a) Express the magnetic flux $\boldsymbol{\Phi}$ through the solenoid in terms of $\boldsymbol{N}$, $\boldsymbol{B}$, $\boldsymbol{S}$, and $\boldsymbol{\theta}$: $\Phi = NBS\cos\theta = NB\pi R^2\cos\theta$.

Substitute numerical values and evaluate $\boldsymbol{\Phi}$:

Φ = 400·(60.0 mT)·3.14·(0.00800 m) 2·cos 600 = 2.41 mWb.

b) Apply Faraday’s law to obtain the ***emf*** induced in the solenoid:

$$\varepsilon_i = -\frac{\Delta\Phi}{\Delta t} = -\frac{0 - 2.41\ \mathrm{mWb}}{1.40\ \mathrm{s}} = 1.72\ \mathrm{mV}.$$

11.7. MUTUAL INDUCTION

If two coils of wire are placed near each other, as in fig. 11.18, a changing current in one will induce an ***emf*** in the other. According to Faraday’s law, the ***emf*** $\boldsymbol{\varepsilon_2}$ induced in coil **2** is proportional to the rate of change of magnetic flux passing through it. This flux is due to the current $\boldsymbol{I_1}$ in coil **1** (which is called the primary coil), and it is often convenient to express the ***emf*** in coil **2** (which is called the secondary coil) in terms of the current in coil **1**.

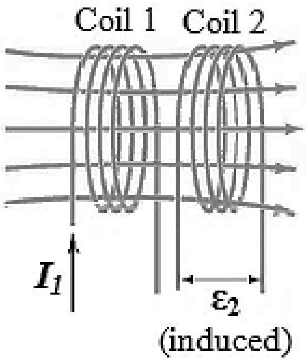


Fig. 11.18. A changing current in one coil will induce a current in the second coil

We let $\boldsymbol{\Phi_{21}}$ be the magnetic flux in each loop of coil **2** created by the current in coil **1**. If coil **2** contains $\boldsymbol{N_2}$ closely wrapped loops, then $\boldsymbol{N_2 \cdot \Phi_{21}}$ is the total flux passing through coil **2**. If the two coils are fixed in space, $\boldsymbol{N_2 \cdot \Phi_{21}}$ is proportional to the current $\boldsymbol{I_1}$ in coil **1**; the proportionality constant is called the ***mutual inductance,*** $\boldsymbol{M_{21}}$, defined by

$$\boldsymbol{M_{21} = \frac{N_2 \Phi_{21}}{I_1}}. \tag{11.12}$$

The ***emf*** $\boldsymbol{\varepsilon_2}$ induced in coil **2** due to a changing current in coil **1** is, by Faraday's law,

$$\boldsymbol{\varepsilon_2 = -N_2 \frac{\Delta \Phi_{21}}{\Delta t}}. \tag{11.13}$$

Let us combine this with equation (11.12) rewritten as $\boldsymbol{\Phi_{21} = M_{21} I_1 / N_2}$ and obtain

$$\boldsymbol{\varepsilon_2 = -M_{21} \frac{\Delta I_1}{\Delta t}}.$$

This relates the change in current in coil **1** to the ***emf*** it induces in coil **2**. The mutual inductance of coil **2** with respect to coil **1**, $\boldsymbol{M_{21}}$, is a "constant" in that it does not depend on $\boldsymbol{I_1}$; $\boldsymbol{M_{21}}$ depends on "geometric" factors such as the size, shape, number of turns, and relative positions of the two coils, and also on whether iron (or some other ferromagnetic material) is present. For example, if the two coils in fig. 11.18 are farther apart, fewer lines of flux can pass through coil **2**, so $\boldsymbol{M_{21}}$ will be less.

Suppose we consider the reverse situation: when a changing current in coil **2** induces an ***emf*** in coil **1**. In this case,

$$\boldsymbol{\varepsilon_1 = -M_{12} \frac{\Delta I_2}{\Delta t}},$$

where $\boldsymbol{M_{12}}$ is the mutual inductance of coil **1** with respect to coil **2**. It is possible to show that $\boldsymbol{M_{12} = M_{21}}$.

Hence, for a given arrangement we do not need the subscripts and we can let $\boldsymbol{M = M_{12} = M_{21}}$, so that

$$\boldsymbol{\varepsilon_1 = -M\frac{\Delta I_2}{\Delta t}} \tag{11.14a}$$

and

$$\boldsymbol{\varepsilon_2 = -M\frac{\Delta I_1}{\Delta t}}. \tag{11.14b}$$

The SI unit for mutual inductance is ***the henry*** (H), where

$$1\ \text{H} = 1\ \text{V·s/A} = 1\ \Omega\text{·s}.$$

Example 11.11.

Two coils have mutual inductance of 1.5 H. If the current in the primary circuit is raised to a value of 50 A in one second after closing the circuit, what is the induced ***efm*** in the secondary?

Solution. Here, mutual inductance of the coils M = 1.5 H. Rate of change of current in the primary is: $\frac{\Delta I}{\Delta t} = \frac{50\ \text{A}}{1\ \text{s}} = 50\ \text{A/s}.$

Then the induced ***efm*** in the secondary is:

$$\varepsilon = M\frac{\Delta I}{\Delta t} = 1.5\ \text{H} \cdot 50\ \text{A}/s = 75\ \text{V}.$$

Example 11.12. **Solenoid and coil.**

A long thin solenoid of length $\boldsymbol{l}$ and cross-sectional area $\boldsymbol{S}$ contains $\boldsymbol{N_1}$ closely packed turns of wire. Wrapped around it is an insulated coil of $\boldsymbol{N_2}$ turns, fig. 11.19. Assume all the flux from coil **1** (the solenoid) passes through coil **2**, and calculate the mutual inductance $\boldsymbol{I_1}$.

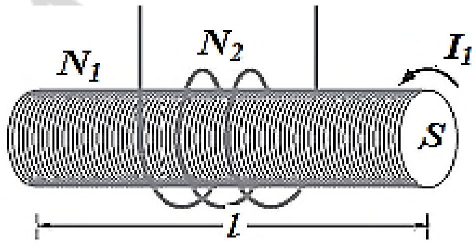


Fig. 11.19. Example 11.12. A long thin solenoid of length $\boldsymbol{l}$ and cross-sectional area $\boldsymbol{S}$ contains $\boldsymbol{N_1}$ closely packed turns of wire. Wrapped around it is an insulated coil of $\boldsymbol{N_2}$ turns

Solution. We first determine the flux produced by the solenoid, all of which passes uniformly through coil $\boldsymbol{N_2}$, using equation for the magnetic field inside the solenoid:

$$B = \mu_0 \frac{N_1}{l} I_1,$$

where $\boldsymbol{n = N_1/l}$ is the number of loops in the solenoid per unit length, and is the current in the solenoid.

The solenoid is closely packed, so we assume that all the flux in the solenoid stays inside the secondary coil. Then the flux Φ_{21} through coil **2** is

$$\Phi_{12} = BS = \mu_0 \frac{N_1}{l} I_1 S.$$

Then the mutual inductance is $M = \dfrac{N_2 \Phi_{21}}{I_1} = \dfrac{\mu_0 N_1 N_2 S}{l}$.

NOTE. *We calculated* $\boldsymbol{M_{21}}$*; if we had tried to calculate* $\boldsymbol{M_{12}}$*, it would have been difficult. Given* $\boldsymbol{M_{12} = M_{21} = M}$*, we did the simpler calculation to obtain* **M.** *Note again that* **M** *depends only on geometric factors, and not on the currents.*

11.8. ENERGY STORED IN A MAGNETIC FIELD

Since an inductor of inductance L in a circuit serves to oppose any change in the current I through it, work must be done by an external source such as a battery in order to establish a current in the inductor. One can conclude that energy can be stored in an inductor. The role played by an inductor in the magnetic case is analogous to that of a capacitor in the electric case. The power P, or rate at which an external ***emf*** ε_{ext} works to overcome the self-induced ***emf*** ε_i and pass current I in the inductor is

$$P_L = \frac{\Delta W_{ext}}{\Delta t} = I \cdot \varepsilon_{ext}.$$

If only the external ***emf*** and the inductor are present, then $\varepsilon_{ext} = -\varepsilon_i$ which implies

$$P_L = \frac{\Delta W_{ext}}{\Delta t} = -I \cdot \varepsilon_i = +IL\frac{\Delta I}{\Delta t}.$$

If the current is increasing with $\Delta I/\Delta t > 0$, then $P > 0$ which means that the external source is doing positive work to transfer energy to the inductor. Thus, the internal energy U_B of the inductor is increased. On the other hand, if the current is decreasing with $\Delta I/\Delta t < 0$, we then have $P < 0$. In this case, the external source takes energy away from the inductor, causing its internal energy to go down. The total work done by the external source to increase the current form zero to I is then:

$$W_{ext} = \frac{1}{2} LI^2.$$

This is equal to the magnetic energy stored in the inductor:

$$W_B = \frac{1}{2} LI^2.$$

The above expression is analogous to the electric energy stored in a capacitor:

$$W_E = \frac{1}{2}\frac{q^2}{C}.$$

We comment that from the energy perspective there is an important distinction between an inductor and a resistor. Whenever a current $\boldsymbol{I}$ goes through a resistor, energy flows into the resistor and dissipates in the form of heat regardless of whether I is steady or time-dependent (recall that power dissipated in a resistor is $\boldsymbol{P_R = IU_R = I^2R}$). On the other hand, energy flows into an ideal inductor only when the current is varying with $\boldsymbol{\Delta I/\Delta t > 0}$. The energy is not dissipated but stored there; it is released later when the current decreases with $\boldsymbol{\Delta I/\Delta t < 0}$. If the current that passes through the inductor is steady, then there is no change in energy since $\boldsymbol{P_L = LI(\Delta I/\Delta t) = 0}$. It makes sense to say there is no energy in inductor with no current.

The energy stored in the magnetic field generated by the current flowing through the inductor can be also given by:

$$\boldsymbol{W = \frac{LI^2}{2} = \frac{I\Phi}{2} = \frac{\Phi^2}{2L},} \tag{11.15}$$

as $\boldsymbol{\Phi = LI}$.

If the current that passes through the inductor is steady, then change in energy since $\boldsymbol{P_L = LI(\Delta I/\Delta t) = 0}$. It makes sense to say there is no energy in inductor with no current.

The energy per unit volume or magnetic energy density is:

$$\boldsymbol{w_B = \text{energy density} = \frac{1}{2}\frac{B^2}{\mu\mu_0},} \tag{11.16}$$

where $\boldsymbol{\mu_0}$ is permeability of free space ($\boldsymbol{\mu_0 = 4\pi \cdot 10^{-7}}$ **T·m/A**), μ is the magnetic permeability of the material.

This equation is analogous to that for an electric field: $\boldsymbol{w_E = \frac{1}{2}\varepsilon\varepsilon_0 E^2}$.

***Example 11.13*. Energy stored in solenoid.**

Calculate the energy associated with the magnetic field of a 200-turn solenoid in which a current of 1.75 A produces a magnetic flux of $3.70 \cdot 10^{-4}$ Wb in each turn.

Solution. The energy $\boldsymbol{W}$ stored in the solenoid when it is carrying a current $\boldsymbol{I}$ is $W = \frac{1}{2}LI^2$. To determine the inductance $\boldsymbol{L}$, it is usually simplest to start with equation $L = \frac{N\Phi}{I}$

$$L = (200)\cdot(3.70\cdot 10^{-4})/1.75 = 4.23\cdot 10^{-2}\ \text{H}.$$

Thus, the energy $\boldsymbol{W}$ stored in the solenoid is:

$$W = \frac{LI^2}{2} = \frac{(4.23\cdot 10^{-2}\,\text{H})\cdot(1.75A)^2}{2} = 0.065\ \text{J}.$$

***Example 11.14*. Energy density.**

Compare the energy density stored in Earth's electric field near its surface to that stored in Earth's magnetic field near its surface.

Solution. We can compare the energy density stored in Earth's electric field to that of Earth's magnetic field by finding their ratio. We'll take Earth's magnetic field to be 0.3 G and its electric field to be 100 V/m.

The energy density in an electric field ***E*** is given by: $w_E = \frac{1}{2}\varepsilon_0 E^2.$

The energy density in a magnetic field ***B*** is given by: $w_B = \frac{1}{2}\frac{B^2}{\mu_0}.$

Express the ratio of $\boldsymbol{w_B}$ to $\boldsymbol{w_E}$ to obtain:

$$\frac{w_B}{w_E} = \frac{B^2}{\mu_0 \varepsilon_0 E^2} = \frac{\left(0.3\ \mathrm{G} \cdot \frac{1\ \mathrm{T}}{10^4\ \mathrm{G}}\right)^2}{(4\pi \cdot 10^{-7}\,\mathrm{N/A^2}) \cdot (8.854 \cdot 10^{-12}\,\mathrm{C^2/N \cdot m^2}) \cdot (100\ \mathrm{V/m})^2} = 8.09 \cdot 10^3.$$

11.9. LC CIRCUIT AND ELECTROMAGNETIC OSCILLATIONS

In any electric circuit, there can be three basic components: resistance ***R***, capacitance ***C***, and inductance ***L***, in addition to a source of emf. Let's consider an ***LC*** circuit, one that contains only a capacitance ***C*** and an inductance, ***L***, fig. 11.20. This is an idealized circuit in which we assume there is no resistance. Let us suppose the capacitor in fig. 11.20, *A* is initially charged so that one plate has charge $\boldsymbol{q_0}$ and the other plate has charge $\boldsymbol{-q_0}$, and the potential difference across it is $\boldsymbol{U = q/C}$. Suppose that at $\boldsymbol{t = 0}$, the switch is closed (fig. 11.20, *B*). The capacitor immediately begins to discharge.

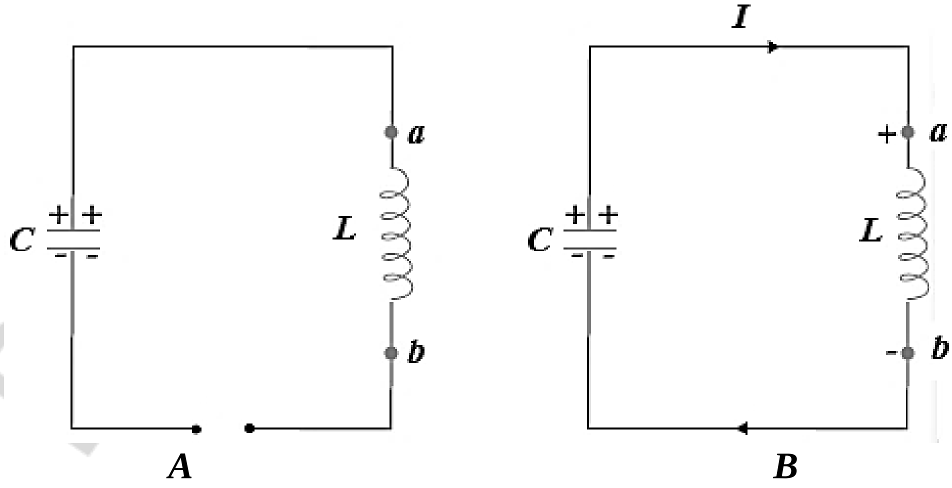


Fig. 11.20. (*A*) an ***LC*** **sircuit**: a capacitor ***C*** be charged $\boldsymbol{q_0}$ and connected to an inductor ***L***. (*B*) The switch is closed in the ***LC*** **circuit**. At the instant shown, the current is increasing so the polarity of induced ***emf*** in the inductor is as shown

The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let $\boldsymbol{q}$ and $\boldsymbol{I}$ be the charge and current in the circuit at time $\boldsymbol{t}$. Since $\boldsymbol{\Delta I/\Delta t}$ is positive, the induced ***emf*** in $\boldsymbol{L}$ will have polarity as shown, i. e. $\boldsymbol{U_b < U_a}$.

We know that a capacitor $\boldsymbol{C}$ and an inductor $\boldsymbol{L}$ can store electrical and magnetic energy, respectively. When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electromagnetic oscillations similar to oscillations in mechanical system.

Let us now try to visualize how this electrical oscillation takes place in the circuit. Fig. 11.21, *a* shows a capacitor with initial charge $\boldsymbol{q_0}$ connected to an ideal inductor. The electrical energy stored in the charged capacitor is

$$W_E = \frac{1}{2}\frac{{q_0}^2}{C} = \frac{1}{2}CU_0^2.$$

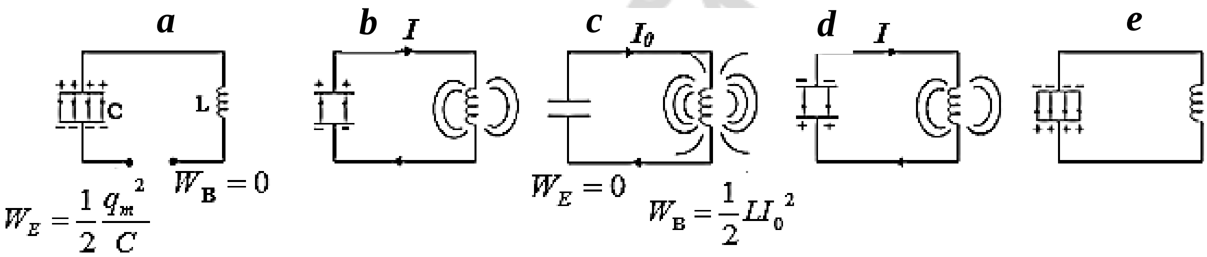


Fig. 11.21. Electromagnetic oscillations in *LC* circuit

Since, there is no current in the circuit, energy in the inductor is zero (W_B = 0). Thus, at times t = 0, t = T/2, t = T and so on (where $\boldsymbol{T} = \mathbf{1/\nu}$ is the period; and angular frequency ($\omega = 2\pi\nu = \sqrt{\frac{1}{LC}}$) all the energy is stored in the electric field of the capacitor:

$$W = W_E = \frac{1}{2}\frac{{q_0}^2}{C}.$$

At t = 0, the switch is closed and the capacitor starts to discharge (fig. 11.21, *b*). As the current increases, it sets up a magnetic field in the inductor and thereby, some energy gets stored in the inductor in the form of magnetic energy:

$$W_B = \frac{1}{2}LI^2.$$

As the current reaches its maximum value $\boldsymbol{I_0}$ (at t = T/4, 3T/4 and so on, as in fig. 11.21, *c*), all the energy is stored in the magnetic field of the inductor (W_E = 0):

$$W = W_B = \frac{1}{2}L{I_0}^2.$$

You can easily check that the maximum electrical energy equals the maximum magnetic energy. The capacitor now has no charge and hence no energy. The current now starts charging the capacitor, as in fig. 11.21, *d*. This process continues till the capacitor is fully charged (at $t = T/2$) (fig. 11.21, *e*). But it is charged with a polarity opposite to its initial state in fig. 11.21, *a*. The whole process just described will now repeat itself till the system reverts to its original state. Thus, the energy oscillates between being stored in the electric field of the capacitor and in the magnetic field of the inductor. The total energy ***W*** is constant, and energy is conserved (fig. 11.22).

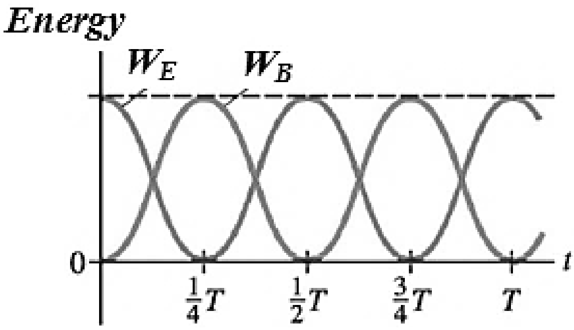


Fig. 11.22. Energy $\boldsymbol{W_E}$ and $\boldsymbol{W_B}$ stored in the capacitor and the inductor as a function of time. Note how the energy oscillates between electric and magnetic. The dashed line at the top is the (constant) total energy: $W = W_E + W_B$

Natural frequency **ν** of oscillation of the ***LC*** circuit is given by:

$$\nu = \frac{1}{2\pi\sqrt{LC}}. \qquad (11.17)$$

At any time the sum of electric and magnetic energy stored in the capacitor and in the inductor has the constant value that is equal to the maximum electrical energy in the capacitor and to the maximum magnetic energy in the inductor:

$$W = \frac{q^2}{2C} + \frac{1}{2}LI^2 = \frac{1}{2}CU_0^2 = \frac{1}{2}LI_0^2.$$

***Example 11.15. LC* circuit.**

A 1200 pF capacitor is fully charged by a 500 V dc power supply. It is disconnected from the power supply and is connected, at $t = 0$, to a 75 mH inductor. Determine: a) the initial charge on the capacitor; b) the maximum current; c) the frequency **ν** and period ***T*** of oscillation; d) the total energy oscillating in the system.

Solution. We use the analysis above, and the definition of capacitance ***q = CU***.

a) The 500 V power supply, before being disconnected, charged the capacitor to a charge of $q_0 = CU = (1.2\cdot10^{-9}\text{ F})\cdot500\text{ V} = 6.0\cdot10^{-7}\text{ C}$.

b) The maximum current, $\boldsymbol{I}_{\mathbf{max}}$, is

$$I_{\max} = \omega q_0 = \frac{q_0}{\sqrt{LC}} = \frac{6.0\cdot 10^{-7}\,\mathrm{C}}{\sqrt{0.075\ \mathrm{H}\cdot(1.12\cdot 10^{-9}\,\mathrm{F})}} = 63\ \mathrm{mA}.$$

c) The frequency $\boldsymbol{\nu}$ can be obtained from: $\nu = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi\sqrt{LC}} = 17\ \mathrm{kHz}$,

and the period $\boldsymbol{T}$ is $T = \dfrac{1}{\nu} = 6.0\cdot 10^{-5}\,\mathrm{s}$.

d) Finally the total energy is $W = \dfrac{q_0^2}{2C} = \dfrac{(6.0\cdot 10^{-7}\,\mathrm{C})^2}{2(1.2\cdot 10^{-9}\,\mathrm{F})} = 1.5\cdot 10^{-4}\,\mathrm{J}$.

PROBLEMS

1. The magnetic field perpendicular to a single 15.6 cm diameter circular loop of copper wire decreases uniformly from 0.550 T to zero. If the wire is 2.05 mm in diameter, how much charge moves past a point in the coil during this operation? (Answer: 4.21 C)

2. What is the mutual inductance of a pair of coils, if a current change of 6 A in one coil causes the flux in the second coil of 2000 turns to change by $12\cdot 10^{-4}$ Wb. (Answer: 0.4 H)

3. A coil that has a self-inductance of 2.00 H and a resistance of 12.0 Ω is connected to an ideal 24.0 V battery. (a) What is the steady-state current? (b) How much energy is stored in the inductor when the steady-state current is established? (Answer: 2.0 A; 4.0 J)

4. An ***LC*** circuit has an inductance of 3.0 mH and a capacitance of 10 μF. Calculate (a) the angular frequency and (b) the period of oscillation. (Answer: 5800 rad/s; 1.1 ms)

5. A capacitor **C** = 25 μF is charged to voltage of 200 V and then discharges on inductance ***L*** = 10 mH. Calculate the whole energy oscillated in the circuit, the maximum value of current and oscillation frequency. (Answer: 0.5 J; 10 A; 318 Hz)

TESTS

1. The unit of magnetic flux in SI system is:

a) Oersted; b) Henry; c) Tesla; d) Weber.

2. The varying magnetic field through a conductor produces electromotive force. This is in accordance with:

a) Faraday's law;
b) Lenz's law;
c) Laplace's law;
d) Ampere's law.

3. A magnet is taken towards a coil and is moved (i) quickly (ii) slowly, then the induced ***emf*** is:

a) larger in first case; b) larger in second case;
c) equal in both cases; d) None of the above.

4. ”The induced ***emf*** is always in such a direction so as to oppose the change that causes it” is called:

a) Lenz’s law; b) Faraday’s law;
c) Kirchoff’s law; d) Joule’s law.

5. The expression for the induced ***emf*** contains a negative sign $\varepsilon_i = -\frac{\Delta\Phi}{\Delta t}$. What is the significance of the negative sign?

a) the induced ***emf*** opposes the changes in the magnetic flux;
b) the induced ***emf*** is produced only when the magnetic flux decreases;
c) the induced ***emf*** is opposite to the direction of the flux;
d) none of above.

6. The induced ***emf*** in a coil rotating in a magnetic field is maximum when the angle between the plane of the coil and direction of the field is:

a) $\pi/4$;
b) zero;
c) $\pi/2$;
d) some angle other than mentioned above.

7. If line of magnetic field is parallel to a surface, then the magnetic flux through the surface is:

a) small but not zero;
b) infinite;
c) zero;
d) large but not infinite.

8. The unit of self-inductance in SI system is:

a) Henry; b) Tesla; c) Weber; d) Oersted.

9. When a conductor of length $\boldsymbol{l}$ is moved perpendicular to a uniform magnetic field $\boldsymbol{B}$ with uniform velocity $\boldsymbol{\upsilon}$, the ***emf*** induced is:

a) $B\,l\,\upsilon$; b) $B\,l/\upsilon$; c) $B\,\upsilon/l$; d) none of these.

10. An airplane is moving north horizontally with a speed of 720 km/h at a place where vertical component of earth’s field is $0.5{\cdot}10^{-4}$ T. What is the induced ***emf*** set up between the tips of the wings 10 m apart?

a) 1.0 V; b) 0.1 V; c) 10 V; d) 0.01 V.

12. GEOMETRICAL OPTICS

12.1. THE RAY MODEL OF LIGHT

In a transparent homogeneous medium light travels in straight lines. This ray model is useful in describing many aspects of light such as reflection, refraction, and the formation of images by mirrors and lenses.

The speed of light in vacuum and in air is $\boldsymbol{c} = \mathbf{3{\cdot}10^8}$ **m/s**. In other transparent materials the light speed $\boldsymbol{v}$ is always less than in vacuum. The ratio of the speed of light in vacuum to the speed $\boldsymbol{v}$ in a given material is called the ***absolute index of refraction n*** of this material:

$$\boldsymbol{n = \frac{c}{\upsilon}}. \qquad (12.1)$$

The refraction index $\boldsymbol{n}$ is different for various materials and it is never less than one. The refractive index of a medium is a measure of light speed in the medium:

$$\boldsymbol{\upsilon = \frac{c}{n}}. \qquad (12.2)$$

The relation between light speeds in two mediums is called the ***relative refractive index*** of the second medium with respect to the first one $\boldsymbol{n_{21}}$:

$$\boldsymbol{n_{21} = \frac{n_2}{n_1} = \frac{\upsilon_1}{\upsilon_2}}. \qquad (12.3)$$

Reflection Law. When light strikes the surface of an object, some of the light is reflected. The rest can be absorbed by the object (and transformed to thermal energy) or transmitted through (in case of transparent medium like glass or water).

When a light ray strikes a flat surface divided two media (fig. 12.1), we define the ***angle of incidence*** $\boldsymbol{\alpha}$ as the angle between an incident ray and the normal (perpendicular) to the surface, and the ***angle of reflection***, $\boldsymbol{\gamma}$ as the angle between the reflected ray and the normal. The ***law of reflection*** is: the incident ray, the normal to the surface and the reflected ray lie in the same plane; the angle of reflection equals to the incidence angle:

$$\boldsymbol{\alpha = \gamma}. \qquad (12.4)$$

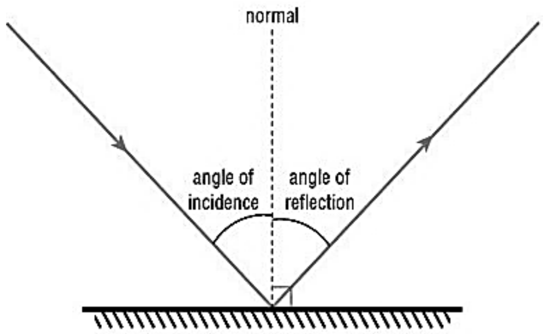


Fig. 12.1. Law of reflection

12.2. IMAGE FORMATION BY A FLAT MIRROR

When light is incident upon a rough surface, even microscopically rough such as this page, it is reflected in many directions, as shown in fig. 12.2. This is called the ***diffuse reflection*** (fig. 12.2). Reflection from a mirror is known as ***specular reflection*** (fig. 12.3) ("Speculum" is Latin for mirror.)

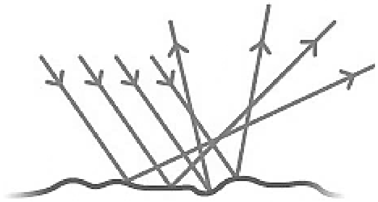


Fig. 12.2. Diffuse reflection

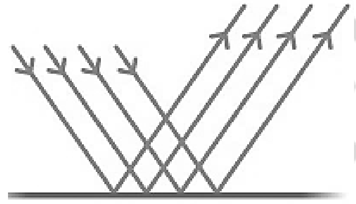


Fig. 12.3. Specular reflection

A flat mirror is one with a smooth flat reflecting surface. Fig. 12.4 shows how an image is formed by a plane mirror according to the ray model. Each ray that reflects from the mirror and enter the eye appear to come from a single point (called the image point) behind the mirror, as shown by the dashed lines. That is, our eyes and brain interpret any ray that enters an eye as having traveled straight-line path. The point from which each ray seems to come is one point on the image. For each point on the object, there is a corresponding image point.

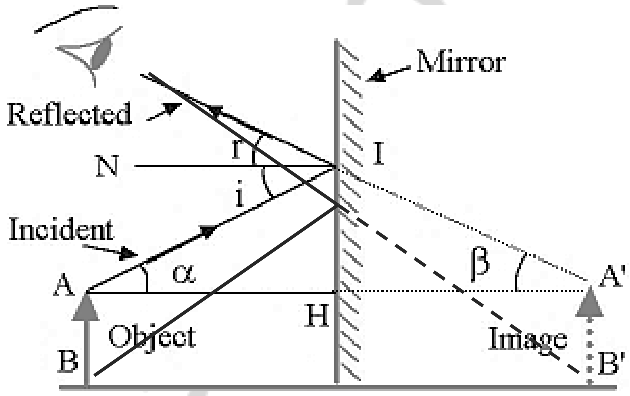


Fig. 12.4. Formation of a virtual image by a plane mirror

The image appears as far behind the mirror as the object is in front. The perpendicular distance from mirror to image (the image distance) is equals the perpendicular distance from object to mirror (the object distance). From the geometry, we can also see that the height of the image is the same as that of the object.

This image would not appear on paper or film placed at the location of the image, therefore it is called a ***virtual*** image. This is to distinguish it from a ***real*** image in which the light does really pass through the image and which therefore could appear on film or on a while sheet of paper or screen placed at the position of the image. Our eyes can see both real and virtual images, as long as the diverging rays enter our pupils.

12.3. REFRACTION. SNELL'S LAW

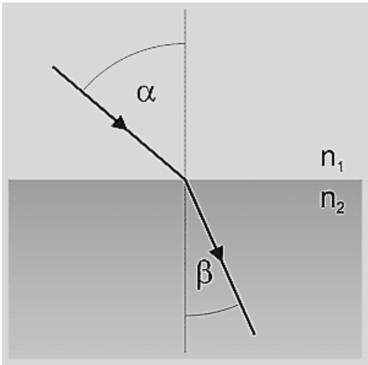


Fig. 12.5. Snell's Law

Refraction is the bending of a light ray when it enters a medium where the refractive index is different (fig. 12.5). The amount of bending depends on the indices of refraction of the two media and is described quantitatively by ***Snell's Law***:

– the incident ray, the normal to the surface and the refrected ray lie in the same plane;

– the ratio of the sine of the angle of incidence **α** to the sine of the angle of refraction **β** is equal to the reciprocal of the ratio of the refractive indices:

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_2}{n_1}, \tag{12.5}$$

where $\boldsymbol{n_1}$ is the absolute index of refraction of the first medium; $\boldsymbol{n_2}$ is the absolute index of refraction of the second medium.

12.4. PHENOMENON OF TOTAL INTERNAL REFLECTION

When the light is travelling from medium with bigger refractive index to the medium with smaller refractive index (for example, from water to air), the angle of refraction is greater than angle of incidence. If the angle of incidence increases, the angle of refraction approaches to 90°. The angle of incidence at which the angle of refraction is equal to 90° is called the ***critical angle*** $\boldsymbol{\alpha_{cr}}$. For angles of incidence greater than $\boldsymbol{\alpha_{cr}}$, there is no refracted ray, all of the incident light is reflected (fig. 12.6). This effect is called ***total internal reflection***. The formula for critical angle $\boldsymbol{\alpha_{cr}}$ is derived from Snell's Law:

$$\sin\alpha_{cr} = \frac{n_2}{n_1}\sin 90° = \frac{n_2}{n_1}. \tag{12.6}$$

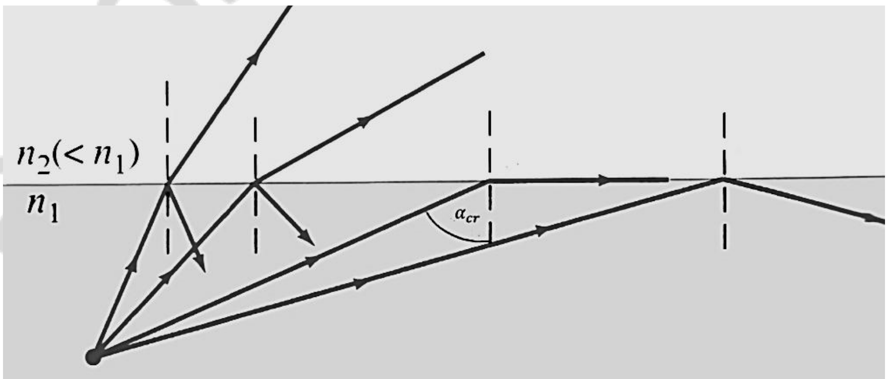


Fig. 12.6. To a total internal reflection

The phenomenon of total internal reflection is used for the fiber optics.

12.5. THIN LENSES. RAY TRACING

A ***thin lens*** is a transparent object with two refracting surfaces. The thickness of lens is negligible compared to radii of curvature of these surfaces. A straight line passed through the curvature centers of these surfaces is the principal axis of the lens.

The lens can be ***convex*** or ***concave***. If the lens is convex, a parallel beam of light passing through the lens is focused to a point on the axis called ***focal point***. In this case the lens is called a converging lens. The focal point placed at the principal axis of the lens is called a ***principal focal point***. The distance of the principal focal point from the center of the lens is called the ***focal length***.

If the lens is concave, a parallel beam of light passing through the lens is diverged. This lens is called a diverging lens. The focal point ***F*** of a diverging lens is defined as the point from which refracted rays seem to be emanating. The distance from ***F*** to the lens is called the focal length, just as for a converging lens (fig. 12.7).

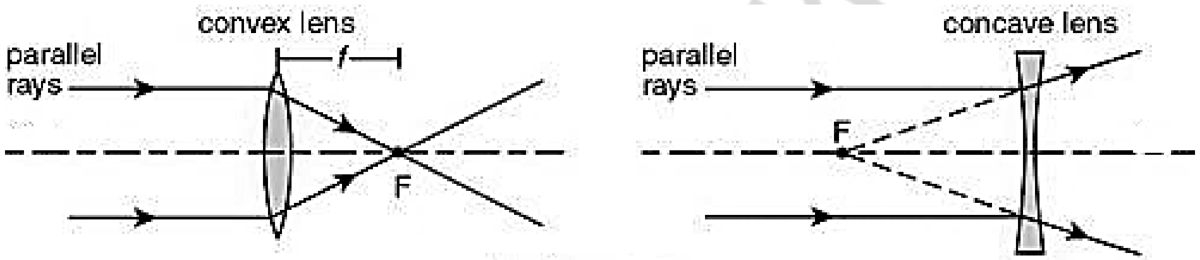


Fig. 12.7. Convex and concave lenses and its principal focuses

The image produced by a converging lens can be constructed using just two of three rays (fig. 12.8):

1. A ray which is parallel to the optical axis refracts through principal focal point behind the lens.
2. A ray which passes through the principal focal point in front of the lens refracts parallel to the optical axis.
3. A ray which passed through the optical center of the lens does not refract at all.

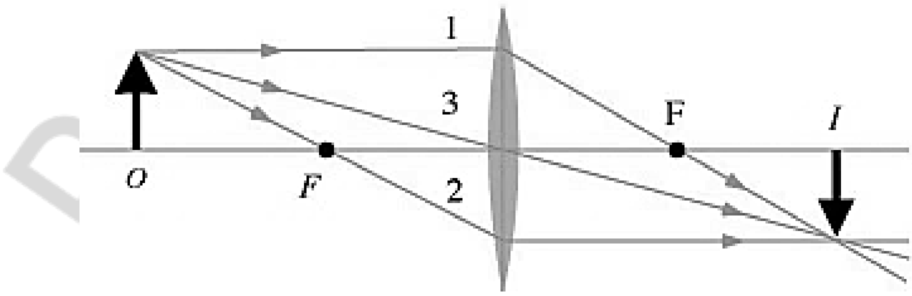


Fig. 12.8. Principal rays for converging lenses

The point where the refracted rays are crossing is the image of the object point. Actually, any two of these rays will suffice to locate the image point, but drawing the third ray can serve as a check.

Converging lens can form both real and virtual image depending on the location of object with respect to focal distance of lens. ***Diverging lens*** can form virtual images only.

For diverging lens the image produced by two of three rays (fig. 12.9):

1) a ray passing through the optical center of the lens;

2) a ray parallel to the principal axis, which refracts through the lens and appears to have come from the principal focus;

3) a ray heading towards the principal focus (on the opposite side of the lens) is refracted by the lens goes parallel to the principal axis.

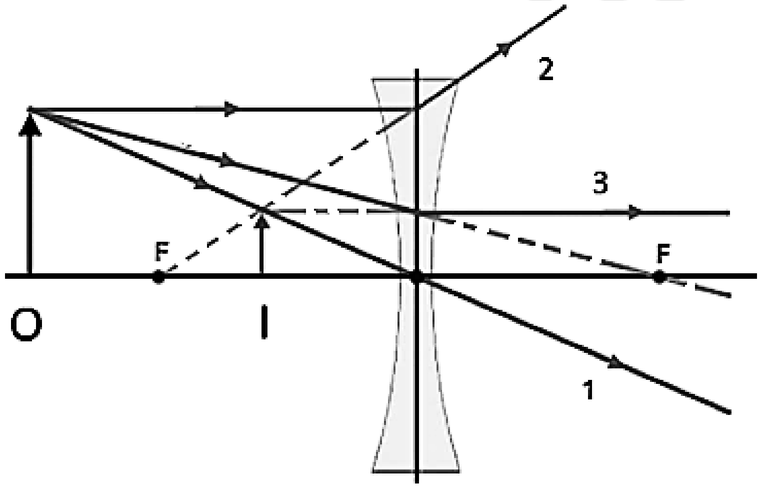


Fig. 12.9. Principal rays for diverging lenses

The real image is located on the opposite side of the lens and it may be projected on a screen or film. The virtual image is located on the same side of the lens as the object and can't be projected on a screen or film. But the eye does not distinguish between real and virtual images — both are visible.

12.6. THE THIN LENS EQUATION. MAGNIFICATION

Optical power is the degree to which a lens converges or diverges light. It is equal to the reciprocal of the focal length of the device:

$$\boldsymbol{D} = \frac{\mathbf{1}}{\boldsymbol{F}}. \qquad (12.7)$$

The shorter the focal length, the stronger the refraction in the lens and the larger the value of the optical power. For converging lenses the optical power is positive, while for diverging lenses it is negative. The most common unit of the optical power measurements is ***diopter*** (***D***): 1 D = 1 m^{-1}.

The ***thin lens equation*** relates the object distance, image distance and focal length:

$$\frac{1}{d}+\frac{1}{f}=\frac{1}{F}. \qquad (12.8)$$

where ***d*** is the distance (measured along the axis) from the object to the lens; ***f*** is the distance (measured along the axis) from the image to the lens; ***F*** is the focal length of the lens (fig. 12.10).

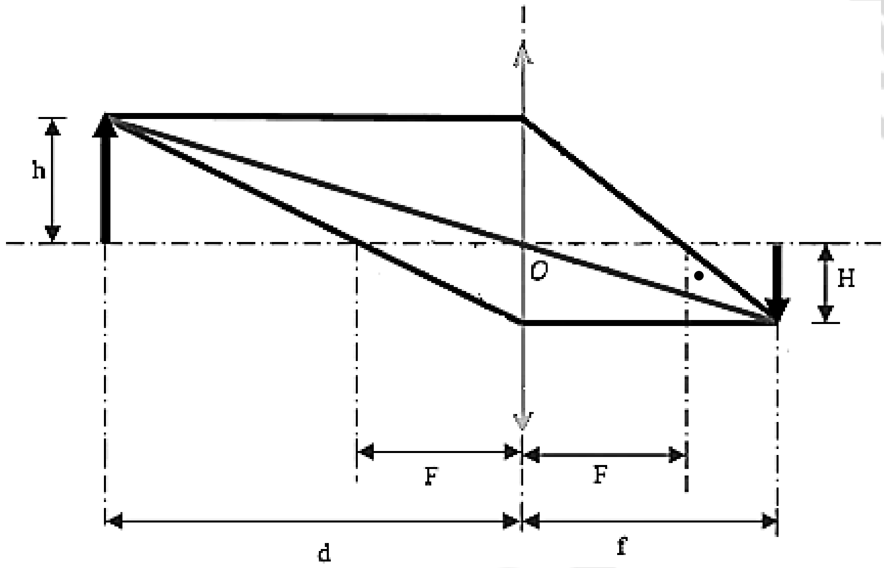


Fig. 12.10. Scheme for the thin lens equation

When using this equation, signs are very important. Distance ***d*** from the object to the lens is always positive. Distance ***f*** from the image to the lens is positive for real images and negative for virtual ones. Focal length ***F*** is positive for converging lenses and negative for diverging ones.

The magnification of the lens is given by:

$$M=\frac{H}{h}=\frac{f}{d}, \qquad (12.9)$$

where ***H*** is a size of an image; ***h*** is a size of an object.

A ***magnifying glass*** (also called a hand lens) is a convex lens that is used to produce a virtual magnified image of an object (fig. 12.11).

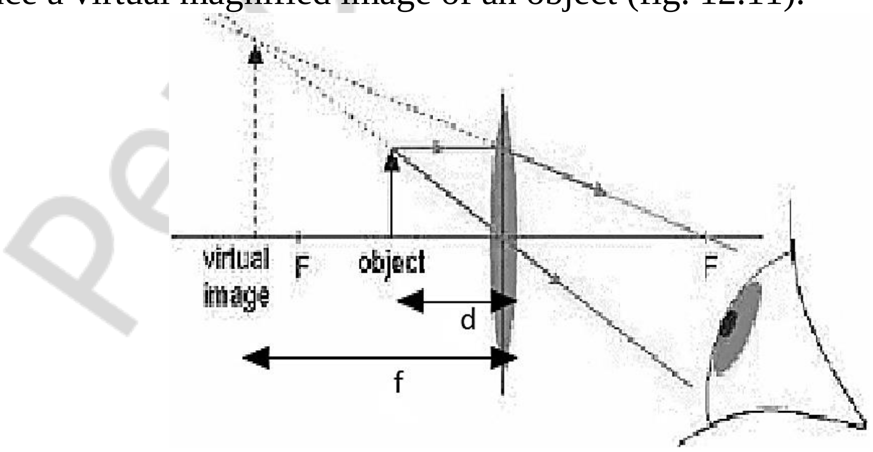


Fig. 12.11. Magnifying glass

Magnification of the magnifying glass can be found as:

$$\boldsymbol{M = \frac{f}{d} = \frac{d_0}{F},} \tag{12.10}$$

where $\boldsymbol{d_0}$ is the distance of the best vision (**$\boldsymbol{d_0}$ = 25 cm**), $\boldsymbol{F}$ is focal length.

Example 12.1.

What is the position and the size of the image of a 7.6-cm-high leaf placed at 1m from a +50-mm-focal-length camera lens?

Solution.

d = 1 m = 100 cm;

F = 50 mm = 5 cm;

f = ?

H = ?

$$\frac{1}{f} = \frac{1}{F} - \frac{1}{d} = \frac{1}{5 \text{ cm}} - \frac{1}{100 \text{ cm}}, \ f = 5.26 \text{ cm}.$$

The magnification is: $M = \frac{H}{h} = \frac{f}{d} = \frac{5.26 \text{ cm}}{100 \text{ cm}} = 0.0526.$

So: $H = M \cdot h = 0.0526 \cdot 7.6$ cm = 0.4 cm.

The image is 4 mm high.

Example 12.2.

An object is placed 10 cm from a 15-cm-focal-length converging lens. Determine the image position and size.

Solution.

d = 10 cm;

F = 15 cm;

f = ?

H = ?

$$\frac{1}{f} = \frac{1}{F} - \frac{1}{d} = \frac{1}{15 \text{ cm}} - \frac{1}{10 \text{ cm}} = -\frac{1}{30 \text{ cm}}, \ f = -30 \text{ cm}.$$

Because $\boldsymbol{f}$ is negative, the image must be virtual and located on the same side of the lens as the object. The magnification:

$$M = \frac{H}{h} = \frac{f}{d} = \frac{-30 \text{ cm}}{10 \text{ cm}} = 3.$$

The image is three times as large as the object and it is upright.

TESTS

1. A light ray has an angle of incidence of 34°. The reflected ray will make what angle with the reflecting surface?

a) 34°; b) 56°; c) 66°; d) 74°.

2. The critical angle for diamond (n = 2.42) submerged in water (n = 1.33) is:

a) 17°; b) 24°; c) 33°; d) 49°.

3. Calculate the index of refraction for a substance in which light travels at $1.97 \cdot 10^8$ m/s.

a) 1.97; b) 0.66; c) 1.42; d) 1.52.

4. The critical angle of zircon is 31°. Which of the following incident angles would result in total internal reflection?

a) 17°; b) 34°; c) 42°;

d) A and C; e) B and C.

5. A object is placed between F and 2F for a diverging lens. The virtual image will be located:

a) between F and 2F;

b) between the lens and F;

c) farther than 2F;

d) there is insufficient information to answer the question.

6. The focal length of a converging lens is 15 cm. An object is placed 40 cm away from the lens. The image will be:

a) smaller and real;

b) larger and real;

c) the same size and real;

d) smaller and virtual;

e) larger and virtual.

PROBLEMS

1. A sharp image is located 373 mm behind a 215-mm-focal-length converging lens. Find the object distance by calculation. (Answer: 508 mm)

2. It is desired to magnify reading material by a factor of 2.5X when a book is placed 9 cm behind a lens. Describe the type of image this would be. What type of lens is needed? What is the power of the lens in diopters? (Answer: 6.7 D upright, magnified; converging lens)

3. An object is located 1.5 m from an 8-D lens. By how much does the image move if the object is moved 0.9 m closer to the lens? (Answer: 0.02 m)

4. What is the focal length of a magnifying glass of 3.8X magnification for a normal eye? (Answer: 6.6 cm)

13. THE WAVE NATURE OF LIGHT

13.1. ELECTROMAGNETIC WAVES SPECTRUM

The ***electromagnetic spectrum*** is distribution of electromagnetic waves according to their wavelength or frequency. The electromagnetic spectrum includes radio waves, infrared, visible and ultraviolet light, ***X***-rays and **γ**-radiation. The wavelength and frequency are different for different type of electromagnetic waves, but a speed of these waves in air and vacuum is constant for all of them and equal to $c = 3 \cdot 10^8 \frac{\text{m}}{\text{s}}$.

Radio waves have wavelengths **λ > 1 mm**; they are used to transmit radio and television signals. ***Infrared (IR)*** is invisible waves with wavelengths **1 mm > λ > 760 nm**. Most of the radiation emitted by heated objects is infrared. ***Visible light*** (**760 nm > λ > 400 nm**) is a small part of the electromagnetic spectrum that human eye can respond. ***Ultraviolet (UV)*** is an electromagnetic radiation with a wavelength from 400 nm to 80 nm, shorter than that of visible light but longer than ***X***-rays. UV radiation is present in sunlight. A small dose of ultraviolet radiation is beneficial to humans, but larger doses cause cataracts and skin cancer. ***X-rays*** have great penetrating power and are used extensively in medical applications as a diagonistic tool. Their wavelength range is **80 nm > λ > 10^{-5} nm**. ***γ-radiation*** have wavelengths of less than **λ < 10^{-5} nm**. They are more penetrating than ***X***-rays. Gamma rays are generated by radioactive atoms.

13.2. LIGHT INTERFERENCE

Light is a transverse, electromagnetic wave that can be seen by humans. An electromagnetic wave consists of oscillating electric and magnetic fields. An electromagnetic wave traveling along an ***x*** axis has an electric field ***E*** and a magnetic field ***B*** with magnitudes that depend on ***x*** and ***t***:

$$\boldsymbol{E} = \boldsymbol{E_0} \sin \omega\left(t - \frac{x}{\upsilon}\right), \quad \boldsymbol{B} = \boldsymbol{B_0} \sin \omega\left(t - \frac{x}{\upsilon}\right), \tag{13.1}$$

$$\boldsymbol{B_0} = \boldsymbol{E_0}/\upsilon \quad \text{or} \quad \boldsymbol{E_0} = \mathbf{B_0} \cdot \upsilon,$$

where E_0 and B_0 are the amplitude values of the electric field strength and the magnetic induction respectively; $\varphi = \omega\left(t - \frac{x}{\upsilon}\right)$ is the phase, $\omega = 2\pi\upsilon$ is the angular frequency; ***t*** is a time; ***υ*** is the velocity, ***x*** is the coordinate.

The wave nature of light was first illustrated through experiments on diffraction and interference.

The interference is superimposition of waves resulting in a steadily in time non uniform distribution of wave energy, with a maxima and minima of light intensity.

To produce the interference of light it is necessary to have two waves or two wave sources with the same frequency and constant phase difference. They are called ***coherent waves*** or ***coherent sources*** correspondently.

Two independent sources of light cannot be coherent because the waves emitted by them would not have the same phase or constant difference in phase. However if the light from the single light source is splited in two beams, these beams may be the coherent waves of light.

For two coherent waves or sources of light the distribution of energy in the surrounding medium is not uniform at all points. There are certain regions, where the intensity of light is maximum. Also, there are other regions, where the intensity of light is minimum. Thus the energy due to sources of light is redistributed. This phenomenon of redistribution of energy in a medium due to superimposition of coherent waves of light is called the ***interference of light***.

At the any point where superimposition of coherent waves occurs the amplitude of oscillator of the electrical field is equal to

$$\boldsymbol{E = \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos(\Delta\varphi)}}, \tag{13.2}$$

where $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ are the electrical amplitude of the first and second waves, $\boldsymbol{\Delta\varphi = (\varphi_1 - \varphi_2)}$ — their phase difference.

If these waves are not coherent their phase difference changes in time very quickly and ($\cos\Delta\varphi$) changes from (–1) to (+1) about 10^9 times per second! Its average value is equal to zero, so the human eye can see the average amplitude only: $\boldsymbol{E = \sqrt{E_1^2 + E_2^2}}$ and average intensity $\boldsymbol{I = I_1 + I_2}$.

But if these waves are coherent, their phase difference is constant in time so the amplitude of resulting oscillation is constant in time too and defined by (13.2).

At the points where $\boldsymbol{\Delta\varphi = 2\pi k}$ $\boldsymbol{(k = 0, \pm1, \pm2 \ldots)}$ $\boldsymbol{\cos\Delta\varphi = +1}$, so the amplitude of an electrical field oscillator will be ***maximum***:

$$\boldsymbol{E_{max} = \sqrt{E_1^2 + E_2^2 + 2E_1E_2} = E_1 + E_2}. \tag{13.3a}$$

Hence the intensity of light at such points becomes ***maximum*** too:

$$\boldsymbol{I = I_1 + I_2 + 2\sqrt{I_1I_2}}.$$

This is called ***constructive interference***.

At some other points, where $\boldsymbol{\Delta\varphi = \pi(2k + 1)}$ ($k = 0, \pm1, \pm2 \ldots$) will be $\boldsymbol{\cos\Delta\varphi = -1}$, so the amplitude of an electrical field oscillators will be ***minimum***:

$$\boldsymbol{E_{min} = \sqrt{E_1^2 + E_2^2 - 2E_1E_2} = |E_1 - E_2|}. \tag{13.3b}$$

Hence the intensity of light at such points is ***minimum***:

$$\boldsymbol{I = I_1 + I_2 - 2\sqrt{I_1I_2}}.$$

This is called ***destructive interference***.

If $\boldsymbol{E_1 = E_2}$ the destructive interference intensity is equal to zero (see 13.3, *b*).

There are no light energy losses at the interference of light. The loss of energy at the points of destructive interference appears as the increase of energy at the points of constructive interference.

A simple experiment of the interference of light was demonstrated by Thomas Young in 1801. It provides solid evidence that light is a wave.

The light from the single light source goes through two slits $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ that serves as a coherent sources. At the screen one has seen the interference pattern called fringes, consisting of alternating light and dark bars, where correspondently maximum and minimum of wave interference has occurred.

The path difference of the two rays coming through two slits to the point $\boldsymbol{M}$ (fig. 13.1) on the screen is:

$$\boldsymbol{\Delta d = d_2 - d_1,} \tag{13.4}$$

and phase difference is:

$$\boldsymbol{\Delta\varphi = \frac{2\pi}{\lambda}\Delta d.} \tag{13.5}$$

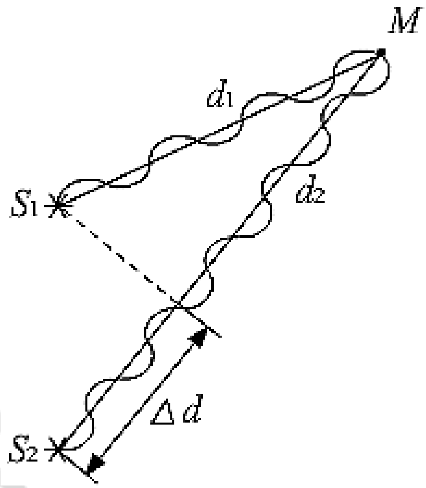


Fig. 13.1. Light interference

If the path difference equals a whole number of wavelengths, then ***constructive interference*** takes place and a bright fringe will appear on the screen:

$$\boldsymbol{\Delta d = k\lambda\ (\Delta\varphi = 2\pi k),\ k = 0, \pm1, \pm2\ \ldots} \tag{13.6}$$

The value of $\boldsymbol{k}$ is called the order of the interference fringe. The first order ($\boldsymbol{k = 1}$), for example, is the first fringe on each side of the central fringe (which is at $\boldsymbol{\Delta\varphi = 0}$, $\boldsymbol{k = 0}$).

If the ray's paths differ by a half number of wavelengths, ***destructive interference*** occurs and dark fringes will appear:

$$\boldsymbol{\Delta d = (2k+1)\frac{\lambda}{2},\ \text{if}\ \ \Delta\varphi = (2k+1)\pi,\ \ k = 0,\ \pm1,\ \pm2\ \ldots} \tag{13.7}$$

In daily life, we observe that a thin layer of an oil spread over water surface and a thin soap film appear colored in sunlight. This production of colors is due to interference of light.

13.3. DIFFRACTION OF LIGHT

The first convincing wave theory for light was established in 1678 by Dutch physicist Christian Huygens. Huygens' wave theory is based on a geometrical construction and allows to tell where a wave front will be at any time in the future if we know its present position. Huygens' principle is:

All points on a wave front serve as point sources of a coherent spherical secondary wavelets. Due to its interference the next new position of the wave front will be a surface tangent to these secondary wavelets.

Huygens' principle is particularly useful for analyzing what happens when waves impinge on an obstacle and the wave fronts are partially interrupted. Huygens' principle predicts that wave bends behind an obstacle.

This phenomenon of bending around obstacles by light waves and its spreading into regions of geometrical shadow of an object is called ***diffraction***. Each element of a wave front starting from a source of light becomes the source of secondary coherent wavelets, which spread out into space. The phenomena of diffraction occurs due to interference of the secondary wavelets that forms a new wave front (fig. 13.2).

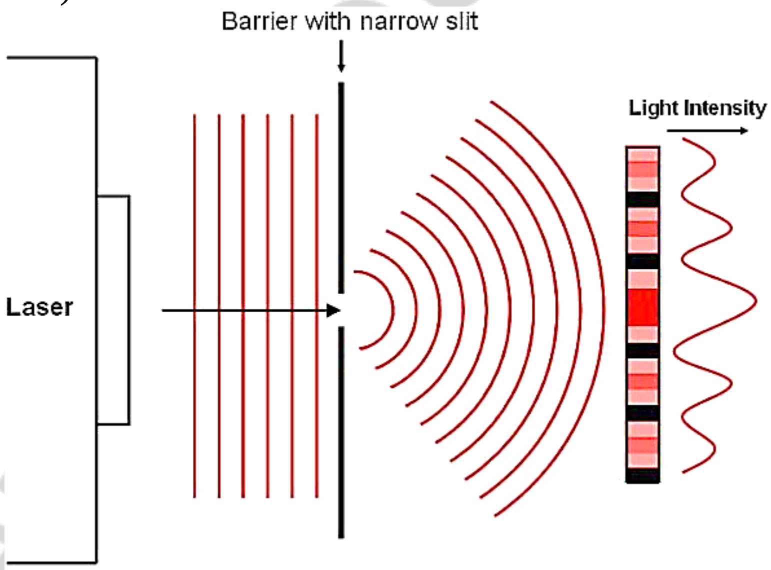


Fig. 13.2. Single slit diffraction

A large number of equally spaced parallel slits is called a ***diffraction grating***. Grating can be made by precision machining of very fine parallel lines on a glass plate. The untouched spaces between the lines serve as the slits. The sum of the slit width ***a*** and the opaque section ***b*** is called the ***period grating d***:

$$\boldsymbol{d = a + b}. \qquad (13.8)$$

We assume parallel rays of light are incident on the grating as shown in fig. 13.3. The slits are narrow enough to produce the coherent diffracted beams, and interference can occur.

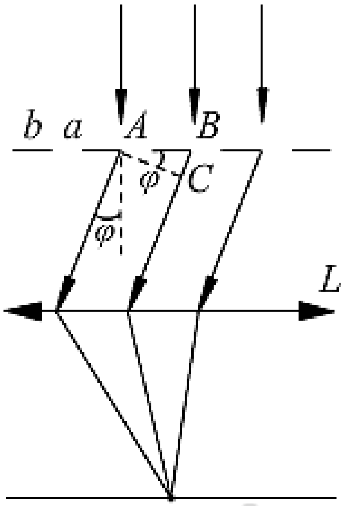


Fig. 13.3. Diffraction grating

Principal maxima diffraction grating occurs under condition:

$$\boldsymbol{d} \sin \varphi = \boldsymbol{k}\lambda,\ \boldsymbol{k} = 0, \pm 1, \pm 2\ \ldots \qquad (13.9)$$

Suppose the light striking a diffraction grating is not monochromatic. Then for all diffraction orders except $\boldsymbol{k} = \mathbf{0}$, each wavelength will produce a maximum at a different angle **φ**. If white light strikes a grating, the central ($\boldsymbol{k} = \mathbf{0}$) maximum will be a sharp white peak. But for all other orders, there will be a distinct color ***spectrum*** (fig. 13.4).

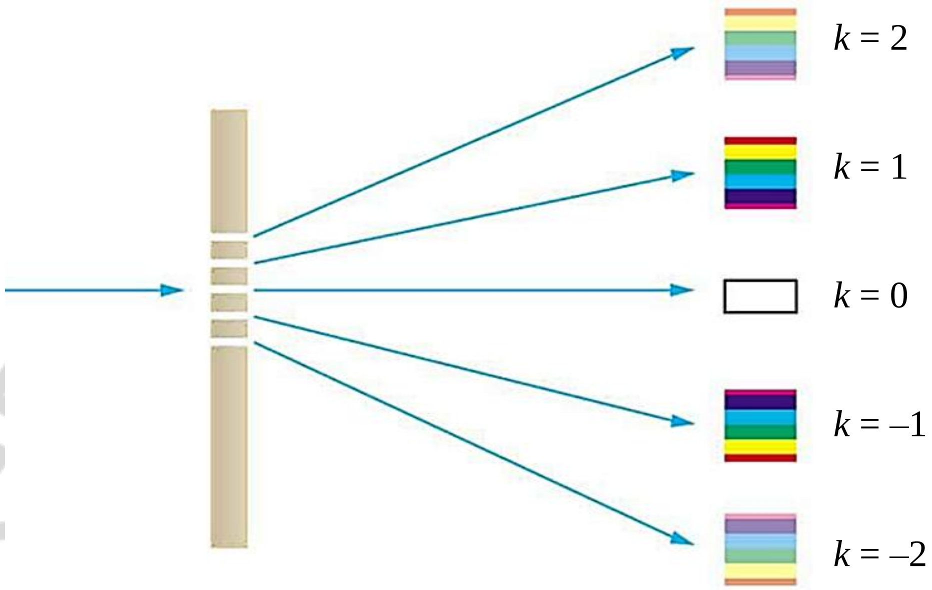


Fig. 13.4. Simplified diagram of diffraction of white light on a diffraction grating

13.4. DISPERSION OF LIGHT

Color is related to the wavelengths or frequencies of the light. Visible light has wavelengths in air in the range of about 400 nm (violet) to 750 nm (red), this is known as the visible spectrum.

White light passing through a prism is separated into different colors (fig. 13.5). The colors in the order of decreasing deviation are violet, indigo, blue, green, yellow, orange and red. The deviation is maximum for violet color and minimum for red color.

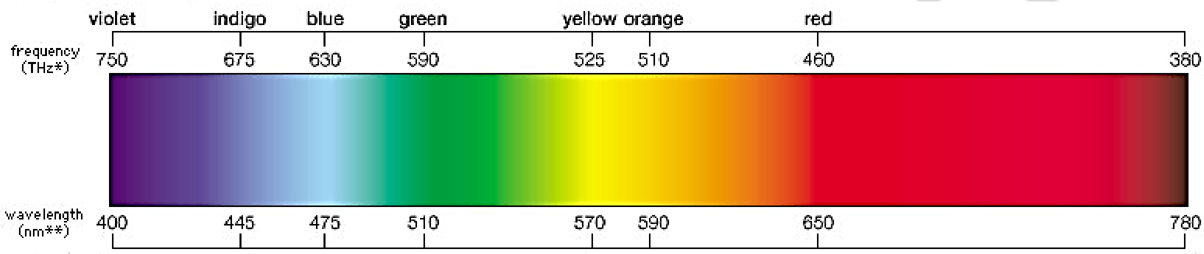


Fig. 13.5. The visible spectrum

The phenomenon of dependence of light speed (or the index of refraction) in a material on the wave frequency (or wavelength) is called ***dispersion of light***.

This phenomenon goes to the splitting of white light into its spectrum, on passing through a prism (fig. 13.6). This happens because the index of refraction of a material depends on the wavelength. White light is a mixture of all visible wavelengths, and when it incident on a prism, the rays of different wavelength are bent to varying degrees. Because usually the index of refraction is greater for the shorter wavelengths, violet light is bent the most and red the least, as indicated. Rainbows are a spectacular example of light dispersion by drops of water.

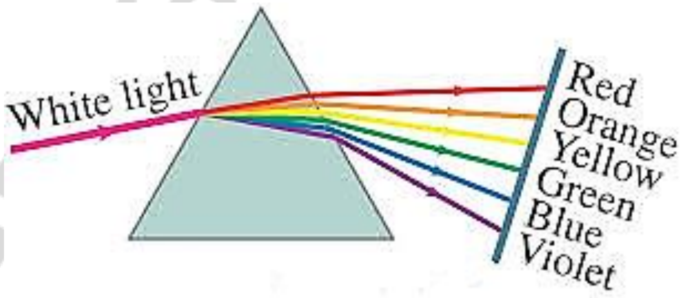


Fig. 13.6. Dispersion of light

As light moves from one medium into another where it travels with a different speed, the ***frequency remains the same***. The wavelength changes as the speed changes. When the light goes from air into a material with index of refraction ***n***, the wavelength becomes:

$$\lambda_n = \frac{\lambda}{n}, \qquad (13.10)$$

where λ is the wavelength in vacuum or air and λ_n is the wavelength in the material with index of refraction n.

TESTS

1. Which of the following types of electromagnetic waves have the longest wavelength:
a) radio waves; b) visible light;
c) infrared rays; d) X-rays.

2. Monochromatic light enters from one medium to the other. Which one of the following properties does not change:
a) amplitude; b) velocity;
c) wavelength; d) frequency.

3. What happens to pattern in Young's experiment when the monochromatic source is replaced by the white light source?
a) all bright fringes become white;
b) all bright fringes get colored from violet to red;
c) only the central fringe is white, all other fringes are colored;
d) no fringes are observed.

4. In Young's experiment the sources should be:
a) incoherent; b) coherent;
c) of any two colors; d) of any frequency.

5. Huygens' principle of secondary waves:
a) allows to find focal length of a thick lens;
b) is a geometrical method to find a wave front;
c) is used to determine the velocity of light.

6. Two sources are said to be coherent if the waves produced by them have the same:
a) wavelength;
b) amplitude;
c) amplitude and wavelength;
d) wavelength and constant phase difference.

7. The bending of light around the corners of an obstacle is called:
a) dispersion; b) refraction;
c) diffraction; d) interference.

8. The dispersion of light in a medium implies that:
a) lights of different wavelengths have the different speeds;
b) lights of different frequencies have the different speeds;

c) refractive indices are different for different wavelengths;
d) all the above.

9. When rays from sun passes through a glass prism, the emergent beam shows all colors of the rainbow. This is due to the phenomenon of:
a) scattering; b) dispersion;
c) diffraction; d) polarization.

10. When white light passes through a glass prism, we get spectrum on the other side of the prism. In the emergent beam the ray which is deviated most is:
a) red ray; b) violet ray;
c) yellow ray; d) green ray.

PROBLEMS

1. The two slits in Young's experiment for producing interference fringes are 0.51 mm apart. Interference fringes of width 0.2 cm are observed on a screen placed 200 cm away from the slits. Find the wavelength of light. (Answer: 510 nm)

2. If a diffraction grating has 600 slits per mm and the second diffraction maximum is observed at 30° of the central maximum, find the wave length of light. (Answer: 417 nm)

3. A diffraction grating has 300 slits per mm, wave length of light is λ = 500 nm. How many diffraction maxima will be observed at the screen? (Answer: 13)

13.5. QUANTUM PROPERTIES OF LIGHT. PHOTOELECTRIC EFFECT

In 1990 German physicist Max Planck proved that light has both particle and electromagnetic wave properties at the same time. A propagation of light can be described by a wave with frequency $\boldsymbol{\nu}$ and wavelength $\boldsymbol{\lambda = \upsilon/\nu}$. And according to the quantum theory light has particle properties in their interaction with matter. Each particle of light is called the ***photon*** (a "particle" concept) and has energy $\boldsymbol{E}$, that relates to wave frequency (a "wave" concept) by

$$\boldsymbol{E = h\nu = h\frac{c}{\lambda}}. \tag{13.10}$$

The proportionality constant $\boldsymbol{h}$ is called Plank's constant. It has the numerical value $\boldsymbol{h = 6.63 \cdot 10^{-34}}$ **J·s** $\boldsymbol{= 4.14 \cdot 10^{-15}}$ **eV·s**. The electron volt (eV) is a small unit of energy:

$$1\ \boldsymbol{eV = 1.6 \cdot 10^{-19}}\ \mathbf{J}.$$

The quantum properties of light have been proved by a photoelectric effect.

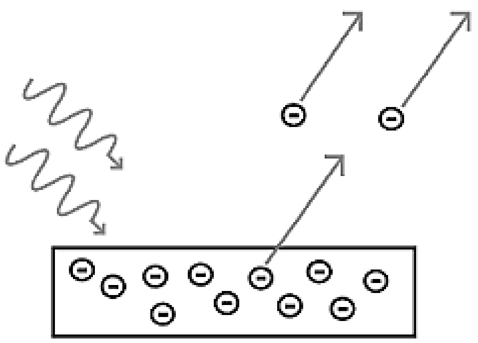


Fig. 13.7. Photoelectric effect scheme

The ***photoelectric effect*** is phenomenon of emission of electrons from substances when light of suitable wavelength fall at them (fig. 13.7). Photoelectric effect occurs in many materials, but it is most easily observed with metals.

This effect can be observed using the device shown in fig. 13.8. A metal plate ***P*** and a smaller electrode ***C*** are placed inside an vacuum glass tube, called a photocell. Electrodes are connected to an ammeter and a battery. When the photocell is in the dark, the ammeter reads zero. But when light of suitable wavelength illuminates the plate, the ammeter indicates a current flowing in the circuit. This is the photoelectric current which is produced when electrons ejected from plate ***C*** reach the plate ***P***.

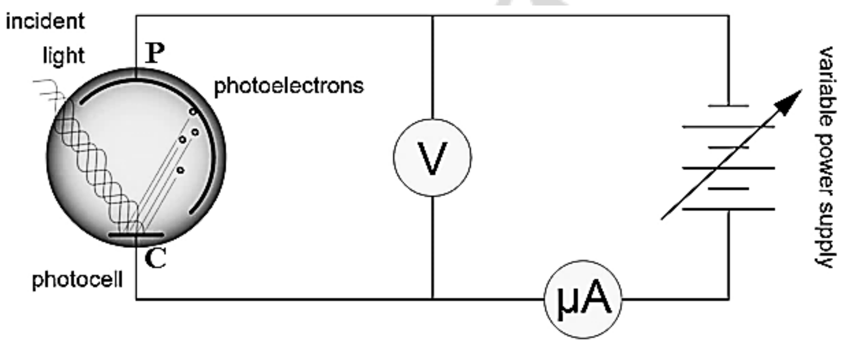


Fig. 13.8. Device for photoelectric effect

The plate ***P*** can be given a positive potential. If this potential is increased, more and more photoelectrons are able to reach ***P***. When all photoelectrons emitted from ***C*** are able to reach the plate ***P*** the photoelectric current is maximum and called the ***saturation current*** (fig. 13.9).

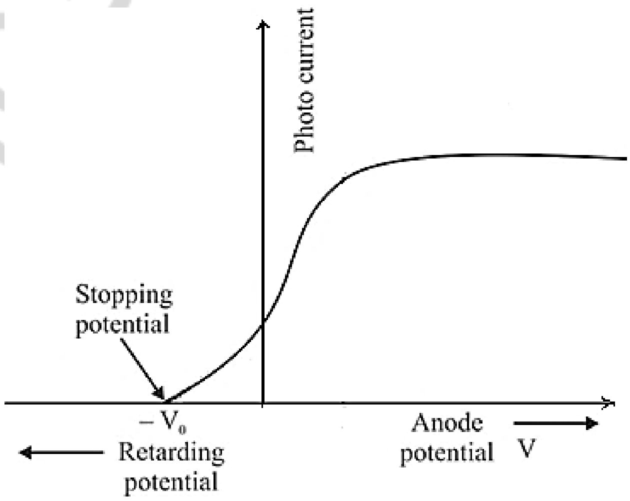


Fig. 13.9. Photoelectric current

If a negative potential is applied to the plate ***P***, the photoelectrons get retarded. As negative potential of ***P*** is increased, more and more electrons are stopped. When the potential of ***P*** is so much negative that even the fastest and most energetic electrons fail to reach ***P***. The photoelectric current will reduce to zero. The minimum negative potential that reduce the photoelectric current to zero is called ***stopping potential***. The photoelectric current will be cut off, when the photoelectron with maximum kinetic energy is stopped:

$$\boldsymbol{eV_0 = \frac{m\upsilon_{max}^2}{2}}, \tag{13.11}$$

where $\boldsymbol{e}$ is charge of electron, $\boldsymbol{V_0}$ is the stopping potential, $\boldsymbol{m}$ is mass of electron, $\boldsymbol{\upsilon_{max}}$ is maximum velocity of electron.

This relation is known as ***Einstein's photoelectric equation***:

$$\boldsymbol{h\nu = \frac{m\upsilon^2}{2} + W_{out}}, \tag{13.12}$$

$\boldsymbol{W_{out}}$ is some minimum energy required just to get an electron out through the surface. $\boldsymbol{W_{out}}$ is called the ***work function (work of electron exit)***. Note that the photoelectric effect does not occur if the frequency is below a certain ***cutoff frequency*** $\boldsymbol{\upsilon_{min}}$ or, equivalently, if the wavelength is greater than the corresponding ***cutoff wavelength***:

$$\boldsymbol{\lambda_{max} = \frac{hc}{W_{out}}}, \tag{13.13}$$

where $\boldsymbol{c}$ is speed of light.

Example 13.1.

Calculate the energy of a photon of blue light, λ = 450 nm in air (or vacuum).

Solution. $E = h\nu = h\dfrac{c}{\lambda} = \dfrac{6.63\cdot 10^{-34}\,\mathrm{J\cdot s}\cdot 3\cdot 10^{8}\,\mathrm{m/s}}{4.5\cdot 10^{-7}\,\mathrm{m}} = 4.4\cdot 10^{-19}\,\mathrm{J}.$

Example 13.2.

What is the kinetic energy and the speed of an electron ejected from a sodium surface whose work function is W_{out} = 2.28 eV when illuminated by light of wavelength (a) 410 nm, (b) 550 nm?

Solution. For λ = 410 nm, the energy of the photons:

$$E = h\nu = h\frac{c}{\lambda} = 4.85\cdot 10^{-19}\,\mathrm{J} = 3.03\ \mathrm{eV}.$$

This energy is greater than W_{out}, then electrons will be ejected with varying amounts of kinetic energy, with a maximum of

$$E_k = h\nu - W_{out} = 3.03\ \mathrm{eV} - 2.28\ \mathrm{eV} = 0.75\ \mathrm{eV} = 1.2\cdot 10^{-19}\ \mathrm{J}.$$

$$\upsilon_{max} = \sqrt{\frac{2E_k}{m}} = \sqrt{\frac{2 \cdot 1.2 \cdot 10^{-19}\,\text{J}}{9.11 \cdot 10^{-31}\text{kg}}} = 5.1 \cdot 10^5 \text{m/s}.$$

For λ = 550 nm, $E = h\nu = h\frac{c}{\lambda} = 3.61 \cdot 10^{-19}\,\text{J} = 2.26$ eV.

Since this photon energy is less than the work function, no electrons are ejected.

TESTS

1. Photoelectric effect reveals the:
a) wave nature of radiation;
b) particle nature of radiation;
c) both wave as well particle nature;
d) none of these.

2. The current in a photocell:
a) decreases with increase of intensity of incident light;
b) increases with increase of intensity of incident light;
c) decreases with increase of frequency of incident light;
d) increases with increase of frequency of incident light.

3. The threshold wavelength for sodium is 500 nm. Its work function is:
a) $W_{out} = 4 \cdot 10^{-19}$ J; b) $W_{out} = 2 \cdot 10^{-19}$ J;
c) $W_{out} = 6 \cdot 10^{-19}$ J; d) $W_{out} = 8 \cdot 10^{-19}$ J.

4. The photoelectric cutoff wavelength of certain metal is 200 nm. The maximum kinetic energy of photoelectrons released by a wavelength of 300 nm is:
a) 2 eV; b) 3 eV; c) 4 eV; d) no electron can be ejected.

5. Photoelectric cell is device which converts:
a) electrical energy into light energy;
b) light energy into electrical energy;
c) electrical energy into sound energy;
d) light energy into elastic energy.

PROBLEMS

1. The work function of metal is 3.45 eV. Calculate what should be the maximum wavelength of light that can eject photoelectrons from the metal. (Answer: 360 nm)

14. ATOMIC PHYSICS

14.1. RUTHERFORD'S MODEL OF ATOM

Rutherford obtained an important insight into the structure of atom by means of performing experiments on the scattering of α-particles on thin gold foil. On the basis of this experiment Rutherford suggested that the atom must consist of a tiny but massive positively charged ***nucleus***, containing over 99.9 % of the mass of the atom, surrounded by ***electrons*** some distance away. The electrons would be moving in orbits about the nucleus — much as the planets move around the Sun — because if they were at rest, they would fall into the nucleus due to electrical attraction (fig. 14.1).

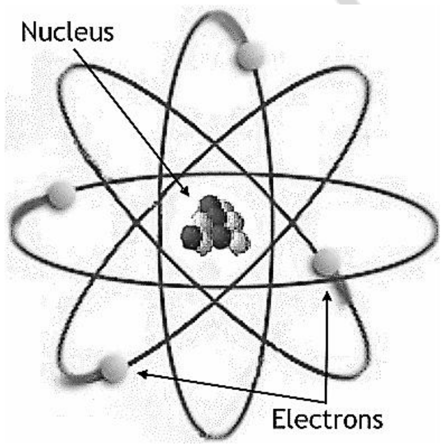


Fig. 14.1. Rutherford's "planetary" model of the atom

Rutherford's experiments suggested that the nucleus must have a radius of about 10^{-15} m. The radius of atoms was estimated to be about 10^{-10} m (if the nucleus were the size of a baseball, the atom would have the diameter of a big city several kilometers across). So an atom would be mostly empty space.

Based wholly on classical physics, the Rutherford model was unable to explain stability of atom. According to the Rutherford model, electrons orbit the nucleus, and since their paths are curved the electrons are accelerating. Hence they should give off light like any other accelerating electric charge. Since light carries off energy and energy is conserved, the electron's own energy must decrease to compensate. Hence electrons would be expected to spiral into the nucleus.

14.2. BOHR'S THEORY OF THE HYDROGEN ATOM

Rutherford's model was replaced in a few years by the Bohr's atomic model, which incorporated some early quantum theory. According to Bohr's atom model the electrons could only orbit the nucleus in particular circular orbits with fixed angular momentum and energy (fig. 14.2). They were not allowed to spiral into the nucleus, because they could not lose energy in a continuous manner; they could only make quantum leaps between fixed energy levels.

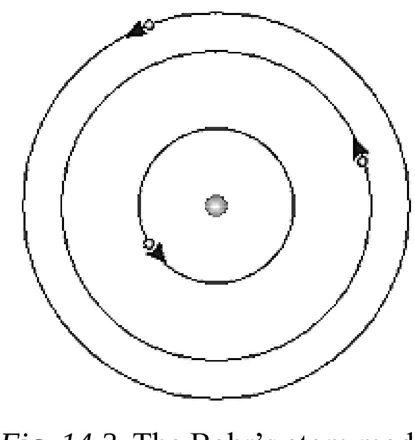


Fig. 14.2. The Bohr's atom model

Bohr's theory of the hydrogenic (one-electron) atom is based on the following ***postulates***:

1. An atom can exist in certain allowed or ***stationary states***, with each state having a definite value for its total energy $E_1, E_2, E_3 \dots E_n$ (fig. 14.3). When the atom is in one of these states it is stable and does not radiate energy.

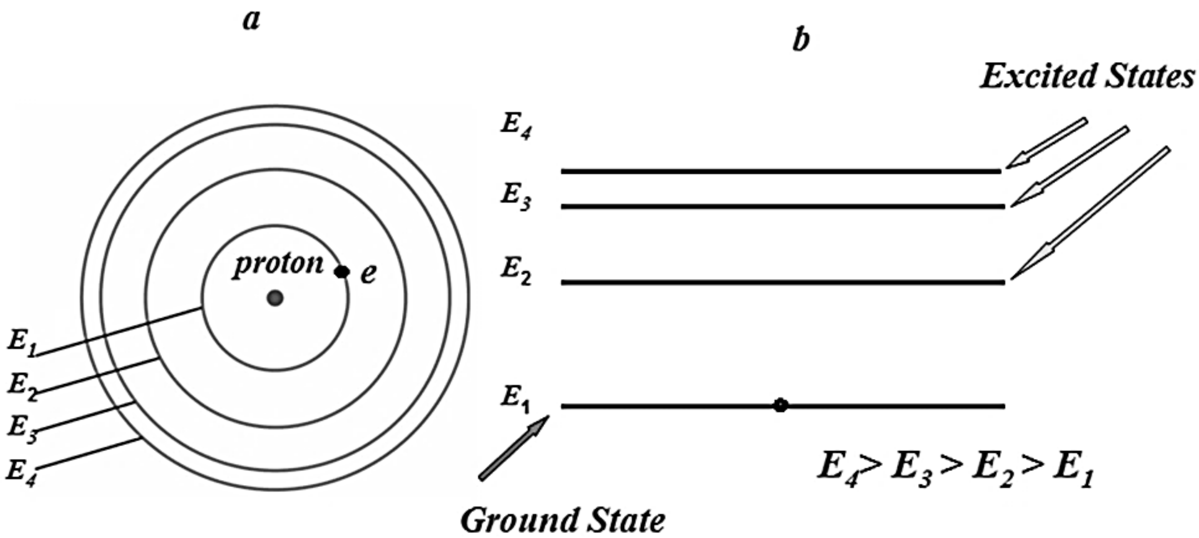


Fig. 14.3. The Bohr atom (*a*). Energy level diagram (*b*)

2. An atom emits or absorbs energy only when an electron moves from one the stable state with energy E_n to another stable state with energy E_k. In a transition from its initial state to its final state, a photon is either ***emitted (if*** $E_n > E_k$***)*** or ***absorbed (if*** $E_n < E_k$ ***)*** and the energy $h\nu$ of the photon is equal to the difference in the energy of the two states:

$$h\nu = |E_n - E_k|. \tag{14.1}$$

Example of the light emission is illustrated in fig. 14.4. The electron jumps from an stationary orbit of higher energy E_2 to an stationary orbit of lower energy E_1 and a photon of energy $h\nu = E_2 - E_1$ is emitted.

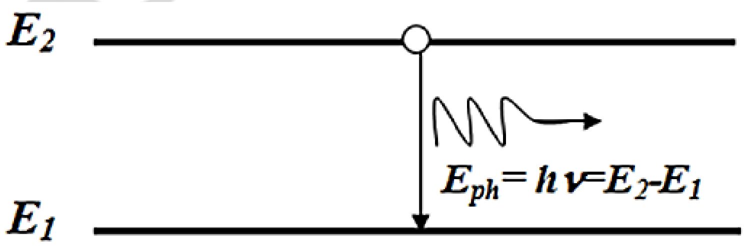


Fig. 14.4. Emission of photon

The electron can absorb energy from some source and jump from a lower energy level to a higher energy level and then emits energy jumping from a higher energy level to a lower energy level as shown in the following fig. 14.5.

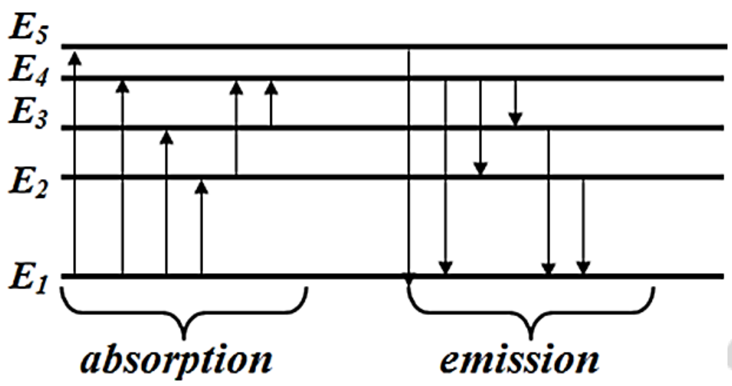


Fig. 14.5. The various ways of an energy absorption and energy emission

Thus from the Bohr model of the atom follows that electrons exist only in the certain energy levels within an atom. The electron energy in these levels has well defined values and electrons jumping between them must absorb or emit the energy equal to the difference between them. The energy emitted as the electron moves to a lower energy level can be in the form of a photon (a particle of light). The wavelength **λ** of the emitted light can be related to its energy:

$$h\nu = \frac{hc}{\lambda} = \Delta E. \qquad (14.2)$$

14.3. ENERGY STATES OF A HYDROGEN ATOM

The above postulates can be used to calculate allowed energies of the atom for different allowed orbits of the electron. The theory developed should be applicable to hydrogen atoms and ions having just one electron. Thus, within the Bohr atom framework, it is valid for He^{+}, Li^{++}, Be^{3+} etc.

Energy states of the hydrogen atom for different allowed orbits of the electron can be described by equation:

$$E_n = \frac{E_0}{n^2} = \frac{-13{,}6\ \text{eV}}{n^2}. \qquad (14.3)$$

where $\boldsymbol{n}$ is the ***principal quantum number***, that labels the orbit radii and also the energy levels, $\boldsymbol{E_0}$ **= –13.6 eV**. The lowest energy level or energy state has energy $\boldsymbol{E_1}$, and is called the ***ground state*****.** The higher states, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$, and so on, are called ***excited states***. The fixed energy levels are also called ***stationary states***.

Notice that although the energy for the larger orbits has a smaller numerical value, all the energies are less than zero. Thus, $\boldsymbol{E_2}$ **= –3.4 eV** is a higher energy than $\boldsymbol{E_1}$ **= –13.6 eV**. Hence the orbit closest to the nucleus ($\boldsymbol{r_1}$) has the lowest energy $\boldsymbol{E_1}$ **= –13.6 eV**. If an electron is free and has kinetic energy, then the total energy $\boldsymbol{E > 0}$. Since $\boldsymbol{E > 0}$ for a free electron, then an electron bound to an atom needs to have $\boldsymbol{E < 0}$. The minimum energy required to remove an electron from an atom initially in the ground state is called the ***binding energy*** or ***ionization energy***. The ionization energy for hydrogen has been measured to be 13.6 eV, and this corresponds precisely to removing an electron from the lowest state, $\boldsymbol{E_1 = -13.6}$ **eV**, up to $\boldsymbol{E = 0}$ where it can be free.

It is useful to show the various possible energy values as horizontal lines on an energy-level diagram. This is shown for hydrogen in fig. 14.6. The electron in a hydrogen atom can be in any one of these levels according to Bohr's theory.

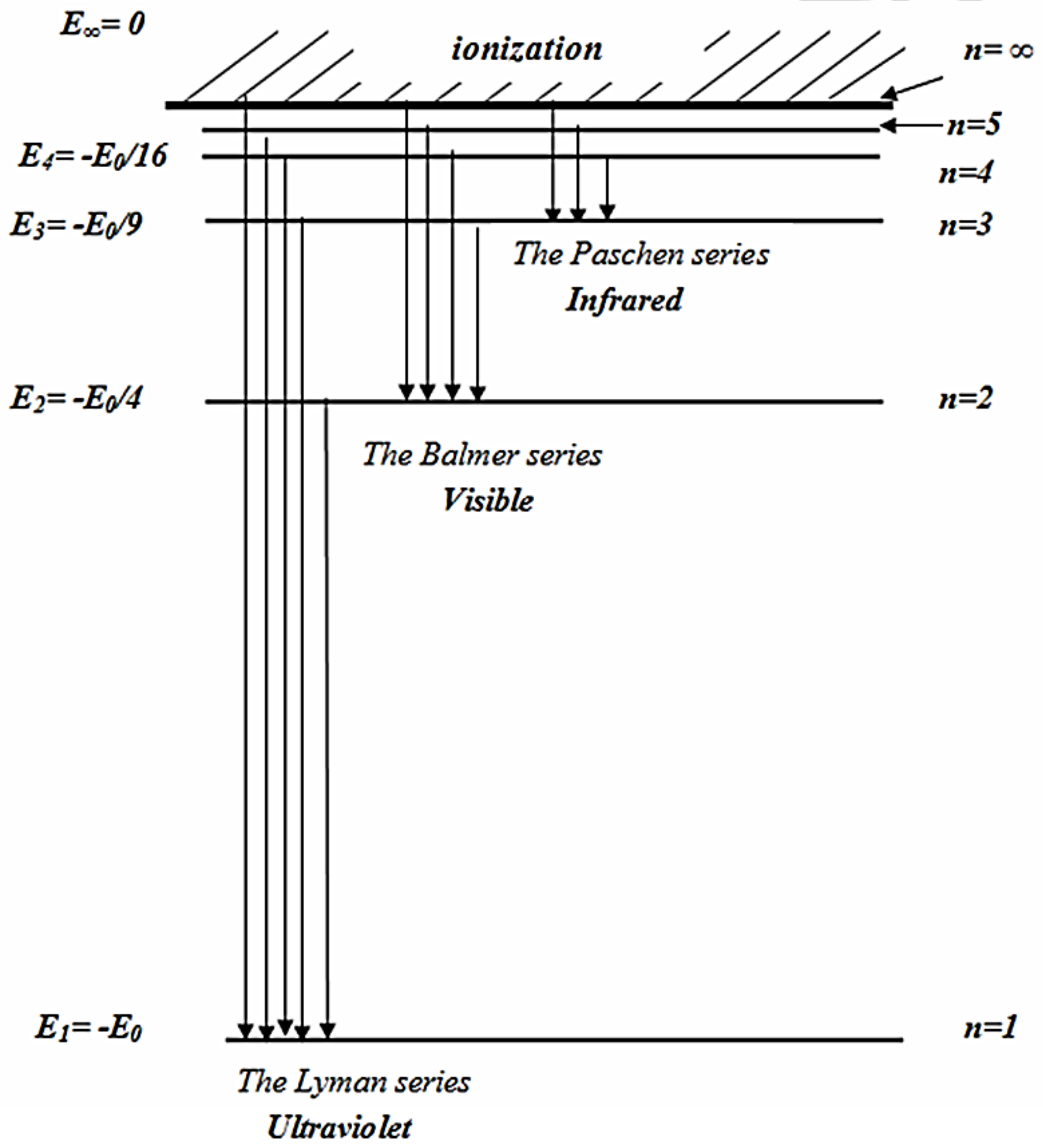


Fig. 14.6. Hydrogen energy diagram illustrating Lyman, Balmer and Paschen series formation

The radiation of atoms that do not interact with one another consists of separate spectral lines. The emission spectrum of atoms is accordingly called a ***line spectrum***. The atomic spectra show the energy structure of atoms therefore the studying of these spectra served as a key to cognition of the structure of atoms. It was noted first of all that the lines in the spectra of atoms are arranged not chaotically, but are combined into groups or, as they are called, ***series of lines***:

$$\boldsymbol{\nu = \frac{E_k - E_n}{h} = \frac{E_0}{h}\left(\frac{1}{n^2} - \frac{1}{k^2}\right),} \tag{14.4}$$

where $\boldsymbol{n = 1, 2, 3, 4,\ldots\ ;\ k = n + 1, n + 2, n + 3\ \ldots}$ Equation (14.4) is the ***generalized Balmer formula***.

Once in an excited state, an atom's electron can jump down to a lower state, and give off a photon in the process. This is, according to the Bohr model, the origin of the emission spectra. The vertical arrows in fig. 14.6 represent the transitions or jumps that correspond to the various observed spectral lines.

The group of spectral lines that corresponds to transitions from any higher energy levels to certain low level forms ***spectral series***. There are some spectral series in hydrogen emission spectrum:

1. ***The Lyman series*** of lines corresponds to transitions or "jumps" that bring the electron down to the ground state $\boldsymbol{E_1}$ ($\boldsymbol{n = 1}$) from any exited energy levels $\boldsymbol{k \geq 2}$ (where $\boldsymbol{n}$ and $\boldsymbol{k}$ are the principal quantum numbers of the states). The lines of the Lyman series are located in ***the ultraviolet range of the spectrum***. The frequencies of the Lyman series are obtained from formula (14.3) if $\boldsymbol{n = 1}$ and $\boldsymbol{k = 2, 3, 4, 5, \ldots}$:

$$\boldsymbol{\nu = \frac{E_0}{h}(1 - \frac{1}{k^2}),} \tag{14.5}$$

where k = 2, 3, 4, 5 …

2. ***The Balmer series*** is characterized by the electron transitions from any exited energy levels $\boldsymbol{k \geq 3}$ to the second energy level $\boldsymbol{E_2}$ **(**$\boldsymbol{n = 2}$**)**, where $\boldsymbol{n}$ and $\boldsymbol{k}$ are the principal quantum numbers of the states. The spectral lines associated with this series are located in ***the visible part of theelectromagnetic spectrum***. The frequencies of the Balmer series can be represented in the form:

$$\boldsymbol{\nu = \frac{E_k - E_2}{h} = \frac{E_0}{h}(\frac{1}{4} - \frac{1}{k^2}),} \tag{14.6}$$

where $\boldsymbol{k}$ = 3, 4, 5, 6 …

3. ***The Paschen series*** is the emission lines corresponding to an electron transitions from $\boldsymbol{k \geq 4}$ to the third energy level $\boldsymbol{E_3}$ **(**$\boldsymbol{n = 3}$**)**. The lines of the Paschen series are located in the ***near infrared range*** of the spectrum. The frequencies of the Paschen series are given by formula:

$$\boldsymbol{\nu = \frac{E_0}{h}(\frac{1}{9} - \frac{1}{k^2}),} \tag{14.7}$$

where $\boldsymbol{k}$ = 4, 5, 6, 7 …

15. PHYSICS OF ATOMIC NUCLEUS

An atomic nucleus consists of elementary particles called ***nucleons***. A ***proton*** is a positively charged particles of mass $1.673 \cdot 10^{-27}$ kg and charge $1.6 \cdot 10^{-19}$ C. A ***neutron*** is electrically neutral and its mass is $1.675 \cdot 10^{-27}$ kg.

A species of nucleus is represented as nuclear symbol ${}^{A}_{Z}\mathbf{X}$, where **X** is the chemical symbol of the element, **Z** is the ***atomic number*** (the number of protons inside the nucleus) and **A** is the ***mass number*** (the number of nucleons). The mass number of the nucleus can be written as:

$$\mathbf{A = Z + N.} \qquad (15.1),$$

where **N** is the number of neutrons.

Nuclei with the same number of protons but different neutron numbers are ***isotopes*** of one another.

The ***nuclear size*** depends on species of nucleus; it grows through the periodic table. The nuclear radius ***R*** and the atomic mass number **A** are related by formula:

$$\boldsymbol{R} = \boldsymbol{R_0} \cdot \sqrt[3]{\mathbf{A}}, \qquad (15.2)$$

where $\boldsymbol{R_0} = \mathbf{1.2 \cdot 10^{-15}\ m}$ — is the radius of hydrogen nuclear (proton).

The ***nuclear charge*** is due to the protons contained in it. Each proton has a positive charge of $\boldsymbol{e_p} = \mathbf{1.6 \cdot 10^{-19}\ C}$. Thus the nuclear charge is equal to:

$$\mathbf{q = Ze}, \qquad (15.3)$$

where **Z** is the atomic number of the nucleus (the amount of protons in nucleus).

The unit of energy commonly used in atomic and nuclear physics is the ***electron volt*** (**eV**):

$$1\ \text{eV} = 1.6 \cdot 10^{-19}\ \text{C} \cdot 1\ \text{V} = 1.6 \cdot 10^{-19}\ \text{J}.$$

Neutrons held together in the nucleus by the strong or ***nuclear forces***. There are extremely short range forces confined only to the nucleus. At short distances, the nuclear forces are much stronger than electrostatic force of repulsion of protons inside the nucleus. Hence the nucleus is stable.

15.1. BINDING ENERGY

The amount of work required to be done to separate the nucleons an infinite distance apart is called the ***binding energy*** of the nucleus. The total mass of a stable nucleus is always less than the sum of the masses of its separate protons and neutrons: $\boldsymbol{M}_{\text{nucleus}} < (\mathbf{Z}_{m_p} + \mathbf{N}_{m_n})$. The difference between the mass of a nucleus and the sum of the masses of protons and neutrons constituting it is called the ***mass defect***:

$$\Delta \boldsymbol{M} = (\mathbf{Z}_{m_p} + \mathbf{N}_{m_n}) - \boldsymbol{M}_{\text{nucleus}}. \qquad (15.4)$$

In formation of any nucleus a certain mass disappears. According to Einstein's theory, this mass defect must be appearing in the form of energy responsible for binding the nucleons together:

$$\boldsymbol{E = \Delta M \cdot c^2 = (Z m_p + N m_n) \cdot c^2 - M_{\text{nucleus}} \cdot c^2.} \qquad (15.5)$$

The binding energy of a nucleus divided by the number of nucleons (or mass number **A**) is binding energy per nucleon:

$$\boldsymbol{\varepsilon = \frac{E}{A}}. \qquad (15.6)$$

Fig. 15.1 shows the binding energy per nucleon as a function of **A** for stable nuclei. The curve rises as **A** increases and reaches a plateau at about 8.7 MeV per nucleon above **A ≈ 40**. Beyond **A ≈ 80**, the curve decreases slowly, indicating that larger nuclei are held together a little less tightly than those in the middle of the Periodic Table.

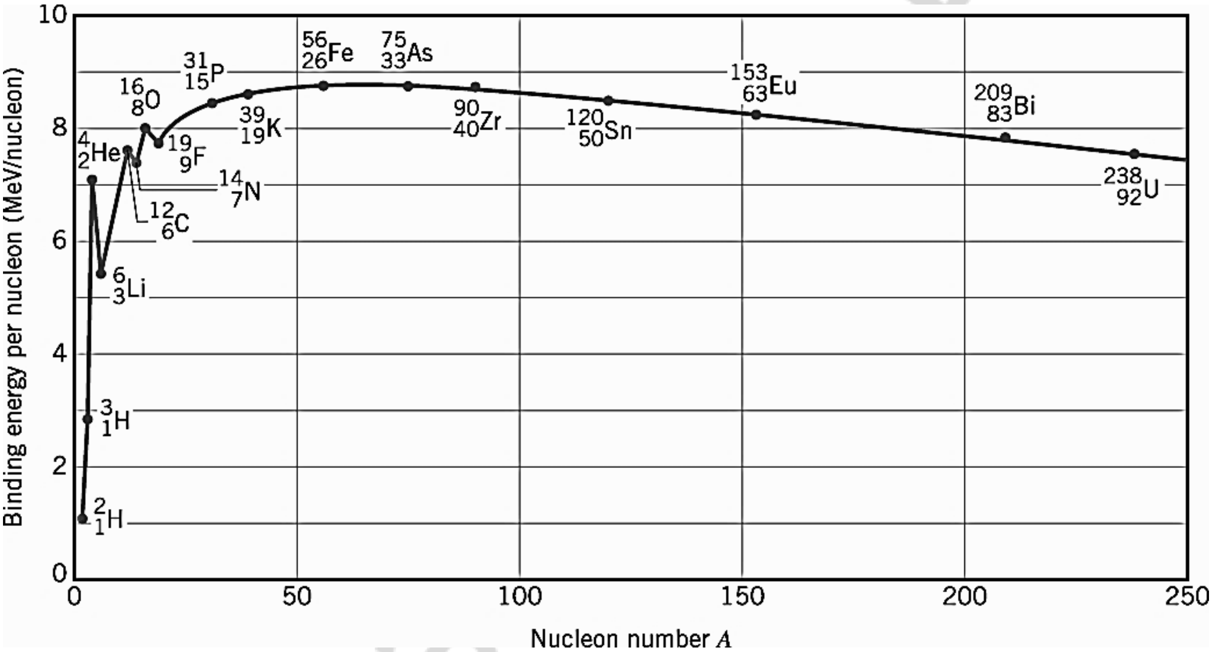


Fig. 15.1. Binding energy per nucleon as a function of mass number A

15.2. NUCLEAR REACTIONS

A ***nuclear reaction*** is the transformation of a stable atomic nucleus into the unstable one by bombarding it with a suitable particle.

A nuclear reaction can be represented symbolically as follows:

$${}^{A}_{Z}X + {}^{4}_{2}He \rightarrow {}^{A+3}_{Z+1}Y + {}^{1}_{1}p + Q. \qquad (15.7)$$

Here the parent nucleus **X** is struck by a helium nucleus to give a daughter nucleus **Y** and proton. In this nuclear reaction **Q** represents the total kinetic energy change in the reaction. It is called ***reaction energy*** or **Q**-value of the reaction. It may be positive or negative. For **Q > 0**, the reaction is said to be exothermic; energy is released in the reaction, so the total kinetic energy is greater after the reaction than before. If **Q** is negative (**Q < 0**), the reaction is said to be endothermic: the final total kinetic energy is less than the initial kinetic energy, and an energy input is required to make the reaction happen.

In any nuclear reaction total energy, electric charge and nucleon number are conserved.

15.3. RADIOACTIVITY. ALPHA, BETA, AND GAMMA RADIATION

Nuclear stability depends on the atomic number **Z** and on the number of neutrons **N**. The light atomic nuclei contain practically as many neutrons as protons (**N/Z = 1**). They are the most stable. In case **N/Z >1.6** the atomic nuclei are unstable and undergo a ***radioactive decay***.

Many unstable isotopes occur in nature, and such radioactivity is called ***natural radioactivity***. Other unstable isotopes can be produced in the laboratory by nuclear reactions; these are said to have ***artificial radioactivity***. Radioactive isotopes are sometimes referred to as radioisotopes or radionuclides.

The most common types of radiation are called alpha, beta, and gamma radiation, named after the first three letters of the Greek alphabet. ***Gamma rays*** are very high-energy photons whose energy is even higher than that of ***X***-rays. ***Beta particles*** are electrons, identical to those that orbit the nucleus, but they are created within the nucleus itself. ***Alpha particles*** are simply the nuclei of helium atoms, that is, every α-particle consists of two protons and two neutrons bound together.

These types of radiation were classified according to their penetrating power. Alpha radiation can barely penetrate a piece of paper. Beta radiation can pass through as much as 3 mm of aluminum. Gamma radiation is extremely penetrating: it can pass through several centimeters of lead and still be detected on the other side.

Each type of radiation has a different charge and hence is bent differently in a magnetic field (fig. 15.3); **α**-rays are positively charged, **β**-rays are negatively charged, and **γ**-rays are neutral.

When a nucleus emits an **α**-particle, the remaining nucleus will be different from the original: it has lost two protons and two neutrons. Alpha decay proceeds according to the following scheme:

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\alpha + \gamma. \tag{15.8}$$

A nucleus that decays spontaneously by emitting an electron or a positron (a positively charged particle with the mass of an electron) is said to undergo beta decay. In beta-minus (β^-) decay, an electron is emitted by a nucleus:

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + {}^{0}_{-1}\beta + \tilde{\nu}. \tag{15.9}$$

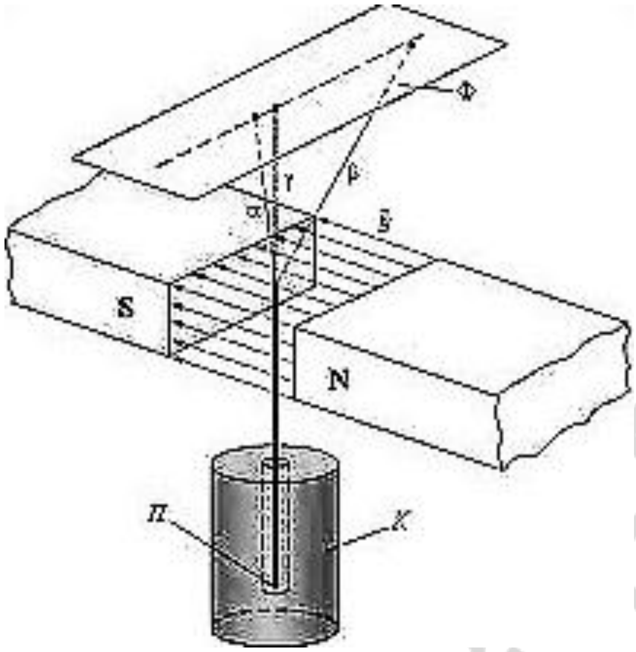


Fig. 15.3. Deflection of alpha, beta and gamma radiation in magnetic field

In beta-plus (β^+) decay, a positron is emitted by a nucleus:

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + {}^{0}_{+1}\beta + \nu. \qquad (15.10)$$

The symbol $\boldsymbol{\nu}$ represents a neutrino, a neutral particle which has a very small mass, that is emitted from the nucleus along with the electron or positron during the decay process; $\tilde{\nu}$ is antineutrino.

Beta-decay is accompanied by the interconversion between neutrons and protons inside a nucleus.

15.4. RADIOACTIVE DECAY LAW

A macroscopic sample of any radioactive isotope consists of a vast number of radioactive nuclei. These nuclei do not all decay at one time. Rather, they decay one by one over a period of time. This is a random process: we can not predict exactly when a given nucleus will decay. But we can determine, on a probabilistic basis, approximately how many nuclei in a sample will decay over a given time period, by assuming that each nucleus has the same probability of decaying in each second that it exists.

Following to the ***radioactive decay law*** the number of undecayed nuclei $\mathbf{N}$ decreases exponentially with time $\boldsymbol{t}$:

$$\mathbf{N = N_0 e^{-\lambda t}}, \qquad (15.11)$$

where $\boldsymbol{\lambda}$ is a constant characteristic of the given radioactive substance and known as the ***decay constant***, $\mathbf{N_0}$ is the initial number of undecayed nuclei at the time $\boldsymbol{t} = \mathbf{0}$.

There are two common time characteristics of how long any given type of radionuclides lasts. One measure is the ***half-life*** $\mathbf{T_{1/2}}$ of a radionuclide, which is the time at which a half of the initial number of nuclei $\mathbf{N_0}$ decays:

$$T_{1/2} = \frac{\ln 2}{\lambda}. \qquad (15.12)$$

The other measure is the ***mean life τ***, which is the time at which $\mathbf{N_0}$ has been reduced to e^{-1} of their initial values:

$$\tau = \frac{1}{\lambda}. \qquad (15.13)$$

15.5. NUCLEAR FISSION

Nuclear fission is a process of breaking up the heavier nuclei into lighter ones with sufficient mass defect, which appears in the form of a tremendous amount of energy. For example, if a massive nucleus like uranium-235 absorbs a low energy (also called thermal) neutron, it breaks apart with the release of energy. This phenomenon was named nuclear fission because of its resemblance to biological fission (cell division).

In a typical ^{235}U fission event, a ^{235}U nucleus absorbs a thermal neutron, producing a compound nucleus ^{236}U in a highly excited state. This nucleus undergoes fission, rapidly emits three neutrons and splits into two fission fragments ^{141}Ba and ^{92}Kr in a typical case:

$$^{235}_{92}\mathrm{U} + {}^{1}_{0}n \rightarrow {}^{236}_{92}\mathrm{U} \rightarrow {}^{141}_{56}\mathrm{Ba} + {}^{92}_{36}\mathrm{Kr} + 3\,{}^{1}_{0}n + 200\,MeV. \qquad (15.14)$$

A tremendous amount of energy is released in a fission reaction because the mass of ^{236}U is considerably greater than the total mass of the fission fragments plus released neutrons.

The energy released in a fission of 1g of $^{235}_{92}\mathrm{U}$ is equal to 82.000 MJ and corresponds to the burning of 3300 kg of coal or 2000 kg of gasoline.

Neutrons released in each fission could be used to create a chain reaction. That is, the three secondary neutrons produced in the reaction bring about the fission of three more $^{235}_{92}\mathrm{U}$ atoms and produce 9 neutrons, which in turn, can bring about the fission of nine $^{235}_{92}\mathrm{U}$ atoms and so on. This process multiplies as shown schematically in fig. 15.3.

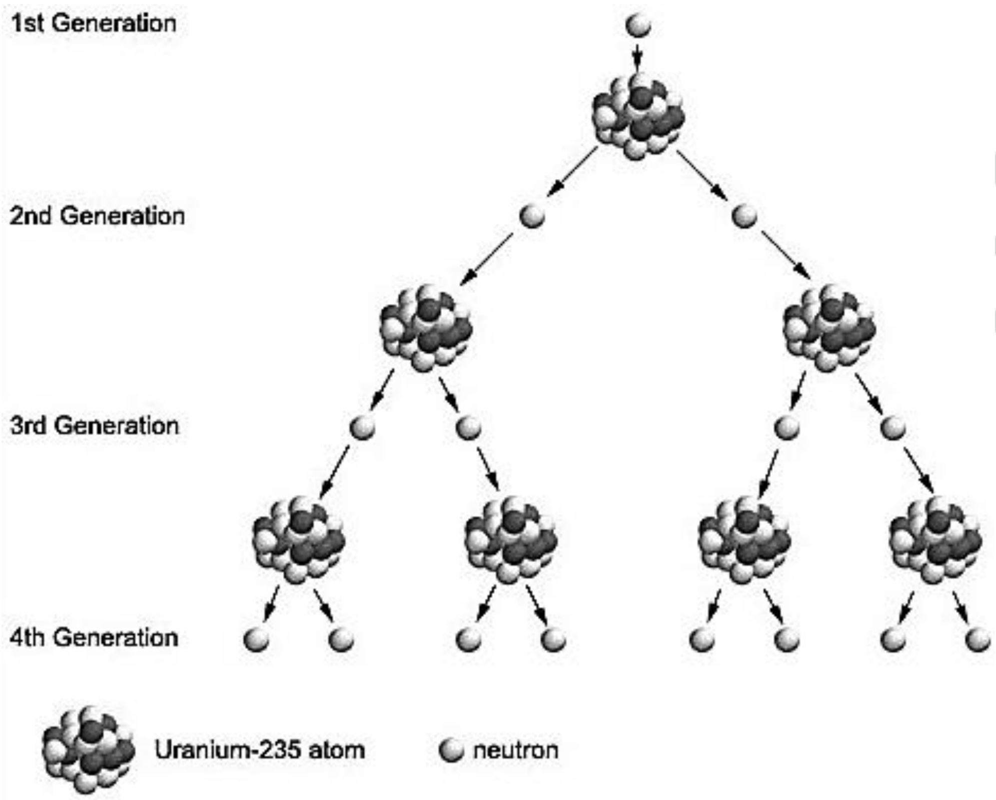


Fig. 15.3. Chain reaction

15.6. ELEMENTARY PARTICLES

Elementary particles are the particles which have no further structure. They are the ultimate building blocks of matter. The various elementary particles have been divided into the following four groups:

1. Photons.
2. Leptons.
3. Mesons.
4. Baryons.

Photons are field particles linked with electromagnetic forces. Every photon is a quantum of radiation with no charge and no rest mass. The energy of a photon is $E = h\nu$. Every photon moves with the velocity of light.

Leptons are particles whose masses are smaller than masses of mesons. Important members of this group are electron, positron and neutrino. Electron is a particle of mass $9.1 \cdot 10^{-31}$ kg. It carries a charge $-1.6 \cdot 10^{-19}$ C, which is taken as a unit negative charge. It is a stable particle. Positron is anti-particle of electron.

Mesons have rest mass 250–1000 times bigger than electron. They are regarded as particles of strong interaction. Most of them owe their existence to cosmic rays.

Baryons have the rest mass equal to or greater than that of proton. The members of this class are proton and neutron.

Tests

1. Rutherford scattering of α-particles by atom shows that:

a) the atom as a whole is positively charged;

b) the atom consists of uniformly distributed positive and negative charged particles;

c) there is no charged particles inside the atom;

d) the atom has a very small positively charged core at the center.

2. In Bohr's model of atom stationary orbits are postulated:

a) in accordance with classical theory of electromagnetism;

b) to meet the condition for dynamic equilibrium of electrons;

c) to meet the condition that the electrons moving in these orbits do not radiate energy;

d) none of the above.

3. According to Bohr's atomic model:

a) electrons radiate energy only when it jumps to another orbit;

b) an atom has heavy, positively charged nucleus;

c) electrons can move only in particular orbits;

d) all of the above statements are true.

4. Two elements having same number of protons but different number of neutrons are called:

a) isobars; b) isotopes; c) isomers; d) isotones.

5. When the electron jumps from orbit n_1 to n_2 orbit, the energy radiated is given by:

a) $h\nu = E_2 - E_1$; b) $h\nu = E_1 - E_2$;
c) $h\nu = E_2 + E_1$; d) $h\nu = E_2/E_1$.

6. The spontaneous emission of high energy particles from the nucleus of the atom is called:

a) radioactivity; b) photoelectricity;
c) thermoelectricity; d) nuclear fusion.

7. The radius of the nucleus is directly proportional to (A = mass number):

a) A^2; b) $\sqrt{A}$; c) $\sqrt[3]{A}$; d) A^3.

8. If radiations from radioactive substance pass through an electromagnetic field, then:

a) all are deflected;

b) only γ-rays are deflected;

c) only α- and β-rays are deflected;

d) only α-rays are deflected.

9. The γ-radiation consists of:
a) photons; b) electrons;
c) protons; d) neutrons.

10. The atomic number 'Z' of the nucleus is:
a) number of protons in it;
b) number of neutrons in it;
c) number of electrons round it;
d) number of deuterons.

11. Alpha-particles consist of:
a) electrons; b) helium nuclei;
c) protons; d) none of these.

12. Beta-particles consist of:
a) high speed moving electrons;
b) protons;
c) neutrons;
d) photons.

13. A nucleus composed of:
a) electrons and protons;
b) neutrons and protons;
c) electrons and neutrons;
d) electrons, protons and neutrons.

14. The nucleus $^{238}_{92}U$ has all the following except:
a) 92 protons; b) 146 neutrons;
c) 238 nucleons; d) 146 electrons.

15. In the reaction: $^{7}_{3}Li + ^{2}_{1}H \rightarrow ^{8}_{3}Li + X$, X is:
a) proton; b) neutron;
c) photon; d) α-particle.

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Fundamental Constants

Quantity	Symbol	Approximate Value
Speed of light in vacuum	c	3.00×10^{8} m/s
Gravitational constant	G	6.67×10^{-11} N·m^2/kq^2
Avoqadro's number	N_A	6.02×10^{23} mol^{-1}
Gas constant	R	8.315 J/mol·K = 1.99 cal/mol·K = = 0.082 atm·liter/mol·K
Boltzmann's constant	k	1.38×10^{-23} J/K
Charge on electron	e	1.60×10^{-19} C
Stefan–Boltzmann constant	σ	5.67×10^{-8} W/m^2·K^4
Permittivity of free space	$\varepsilon_0 = (1/c^2\mu_0)$	8.85×10^{-12} C^2/N·m^2
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ T·m/A
Planck's constant	h	6.63×10^{-34} J·s
Electron rest mass	m_e	9.11×10^{-31} kg = 0.000549 u = = 0.511 MeV/c^2
Proton rest mass	m_p	1.6726×10^{-27} kg = 1.00728 u = = 938.3 MeV/c^2
Neutron rest mass	m_n	1.6749×10^{-27} kg = 1.008665 u = = 939.6 MeV/c^2
Atomic mass unit (1 u)		1.6605×10^{-27} kg = 931.5 MeV/c^2

Other Useful Data

Joule equivalent (1 cal)	4.186 J
Absolute zero (0 K)	–273.15 °C
Earth: Mass Radius (meam)	5.97×10^{24} kg 6.38×10^{3} km
Moon: Mass Radius (meam)	7.35×10^{22} kg 1.74×10^{3} km
Sun: Mass Radius (meam)	1.99×10^{30} kg 6.96×10^{5} km
Earth-sun distance (meam)	149.6×10^{6} km
Earth-moon distance (meam)	384×10^{3} km

The Greek Alphabet

Alpha	Α	α	Nu	Ν	ν
Beta	Β	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	Ο	ο
Delta	Δ	δ	Pi	Π	π
Epsilon	Ε	ε	Rho	Ρ	ρ
Zeta	Ζ	ζ	Sigma	Σ	σ
Eta	Η	η	Tau	Τ	τ
Theta	θ	θ	Upsilon	Υ	υ
Iota	Ι	ι	Phi	Φ	ϕ, φ
Kappa	Κ	κ	Chi	Χ	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	Μ	μ	Omega	Ω	ω

Mathematical Signs and Symbols

$\propto$	is proportional to	$\leq$	is less than or equal to
$=$	is equal to	$\geq$	is greater than or equal to
$\approx$	is approximately equal to	Σ	sum of
$\neq$	is not equal to	x	average value of x
$>$	is greater than	Δx	change in x
$\gg$	is much greater than	$\Delta x \to 0$	Δx approaches zero
$<$	is less than	n!	n (n – 1) (n – 2) … (1) of
$\ll$	is much less than		

SI Derived Units and Their Abbreviations

Quantity	Unit	Abbreviation	In Terms of Base Units
Force	newton	N	$kg{\cdot}m/s^2$
Energy and work	joule	J	$kg{\cdot}m^2/s^2$
Power	watt	W	$kg{\cdot}m^2/s^3$
Pressure	pascal	Pa	$kg/(m{\cdot}s^2)$
Frequency	hertz	Hz	S^{-1}
Electric charge	coulomb	C	$A{\cdot}s$
Electric potential	volt	V	$kg{\cdot}m^2/(A{\cdot}s^3)$
Electric resistance	ohm	Ω	$kg{\cdot}m^2/(A^2{\cdot}s^3)$
Capacitance	farad	F	$A^2{\cdot}s^4/(kg{\cdot}m^2)$
Magnetic field	tesla	T	$kg/(A{\cdot}s^2)$
Magnetic flux	weber	Wb	$kg{\cdot}m^2/(A{\cdot}s^2)$
Inductance	henry	H	$kg{\cdot}m^2/(s^2{\cdot}A^2)$

Metric (SI) Multipliers

Prefix	Abbreviation	Value
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deka	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

Elastic moduli

Material	Young’s modulus, *E* (N/m^2)	Shear modulus, *G* (N/m^2)
	Solids	
Iron, cast	100×10^9	40×10^9
Steel	200×10^9	80×10^9
Brass	100×10^9	35×10^9
Aluminum	70×10^9	25×10^9
Concrete	20×10^9	
Brick	14×10^9	
Marble	50×10^9	
Granite	45×10^9	
Wood (pine)		
(parallel to grain)	10×10^9	
(perpendicular to grain)	1×10^9	
Nylon	5×10^9	
Bone (limb)	15×10^9	80×10^9

Densities of Substances*

Substance	Density, *p* (kg/m^3)
Solids	
Aluminum	2.70×10^3
Iron and steel	7.8×10^3
Copper	8.9×10^3
Lead	11.3×10^3
Gold	19.3×10^3
Concrete	2.3×10^3
Granite	2.7×10^3
Wood (typical)	$0.3–0.9 \times 10^3$
Glass, common	$2.4–2.8 \times 10^3$
Ice	0.917×10^3
Bone	$1.7–2.0 \times 10^3$
Liquids	
Water (4 °C)	1.00×10^3
Sea water	1.025×10^3
Blood, plasma	1.03×10^3
Blood, whole	1.05×10^3
Mercury	13.6×10^3
Alcohol, ethyl	0.79×10^3
Gasoline	0.68×10^3
Gases	
Air	1.29
Helium	0.179
Carbon dioxide	1.98
Water (steam) (100 °C)	0.598

∗ Densities are given at 0 °C and 1 atm pressure unless otherwise specified.

Latent Heats (at 1 atm)

Substance	Melting point (°C)	Heat of Fusion		Boiling Point (°C)	Heat of Vaporization	
		kcal/kg*	J/kg		kcal/kg*	J/kg
Water	0	79.7	3.33×10^5	100	539	22.6×10^5
Lead	327	5.9	0.25×10^5	1750	208	8.7×10^5
Silver	961	21	0.88×10^5	2193	558	23×10^5
Iron	1808	69.1	2.89×10^5	3023	1520	63.4×10^5
Tungsten	3410	44	1.84×10^5	5900	1150	48×10^5

∗ Numerical values in kcal/kg are the same in cal/g.

Resistivity and Temperature Coefficients (at 20 °C)

Material	Resistivity, ρ (Ω·m)	Temperature Coefficients, α $(°C)^{-1}$
Conductors		
Silver	1.59×10^{-8}	0.0061
Copper	1.68×10^{-8}	0.0068
Gold	2.44×10^{-8}	0.0034
Aluminum	2.65×10^{-8}	0.00429
Tungsten	5.6×10^{-8}	0.0045
Iron	9.71×10^{-8}	0.00651
Platinum	10.6×10^{-8}	0.003927
Mercury	98×10^{-8}	0.0009
Nichrome (alloy of Ni, Fe, Cr)	100×10^{-8}	0.0004
Semiconductors∗		
Carbon (graphite)	$(3–60) \times 10^{-5}$	–0,0005
Germanium	$(1–500) \times 10^{-3}$	–0,05
Silicon	0,1–60	–0,07
Insulators		
Glass	$10^{9}–10^{12}$	
Hard rubber	$10^{13}–10^{15}$	

∗ Values depend strongly on presence of even slight amounts of impurities.

Indices of refraction*

Material	$n = \frac{c}{v}$
Vacuum	1.0000
Air (at STP)	1.0003
Water	1.33
Ethyl alcohol	1.36
Glass	
Fused quartz	1.46
Crown glass	1.52
Light flint	1.58
Lucite or Plexiglas	1.51
Sodium chloride	1.53
Diamond	2.42

∗ λ = 589 nm.

Rest Masses in Kilograms, Unified Atomic Mass Units, and MeV/c^2

Object	Mass		
	kg	u	MeV/c^2
Electron	9.1094×10^{-31}	0.00054858	0.51100
Proton	1.67262×10^{-27}	1.007276	938.27
${}^1_1$H atom	1.67353×10^{-27}	1.007825	938.78
Neutron	1.67493×10^{-27}	1.008665	939.57

The Types of Radioactive Decay

α *decay*:

$$ {}^{A}_{Z}N \to {}^{A-4}_{Z-2}N' + {}^{4}_{2}He $$

β *decay*:

$$ {}^{A}_{Z}N \to {}^{A}_{Z+1}N' + e^- + \tilde{\nu} $$

$$ {}^{A}_{Z}N \to {}^{A}_{Z-1}N' + e^+ + \nu $$

$$ {}^{A}_{Z}N + e^- \to {}^{A}_{Z-1}N' + e^- + \nu \text{ [EC]}^* $$

$$ {}^{A}_{Z}N^{**} \to {}^{A}_{Z}N + \gamma $$

* Electron capture; ** indicates the excited state of a nucleus.

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